

Totally equimodular matrices: decomposition and triangulation

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Université Libre de Bruxelles¹

Joint work with: Patrick Chervet and Roland Grappe

LIGM Seminar – Tuesday 13 April



Section 1

Context

Linear optimization problem

Let A be a matrix of size $m \times n$ and b be a vector.

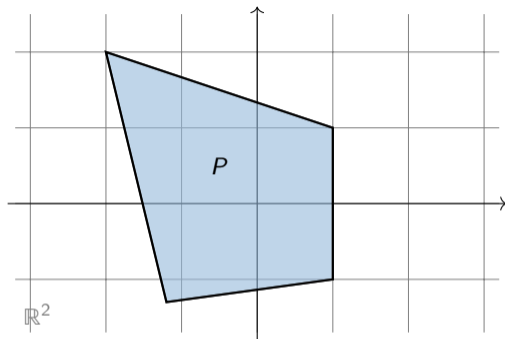
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Feasible set of $Ax \leq b$:

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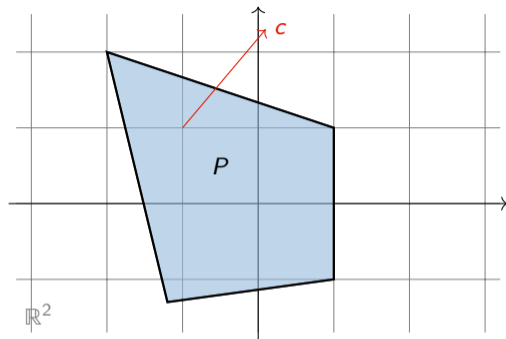
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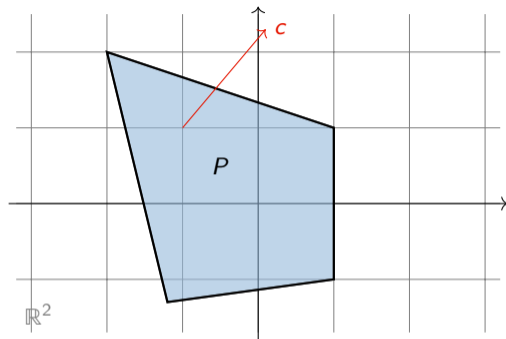
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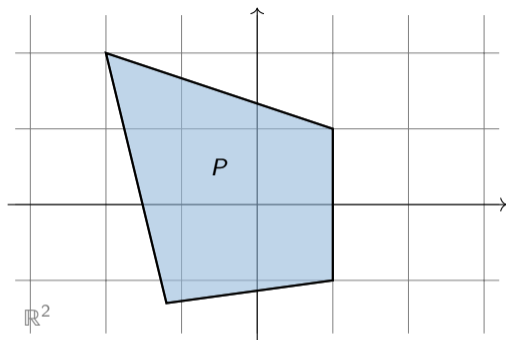
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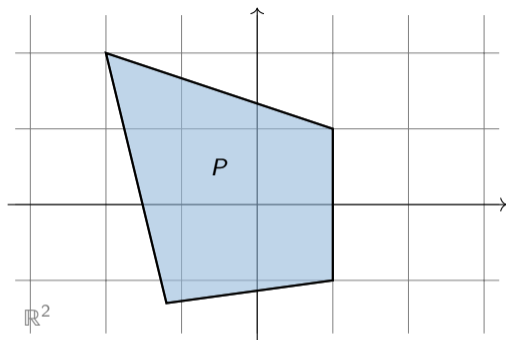
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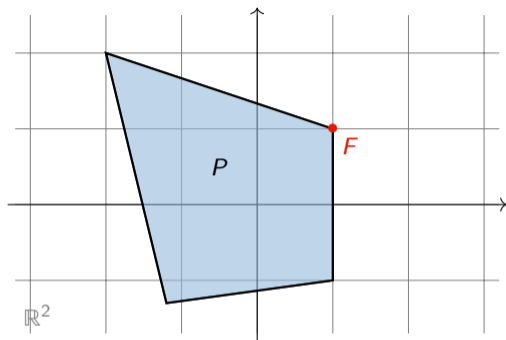
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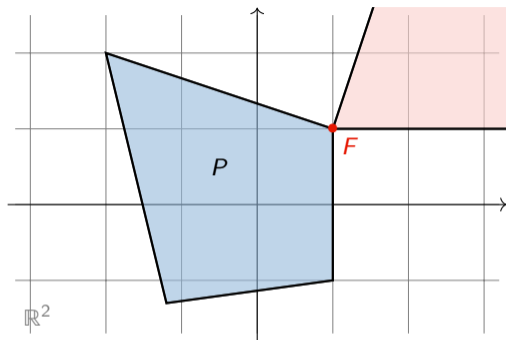
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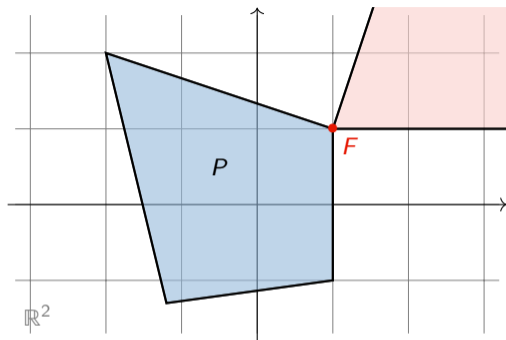
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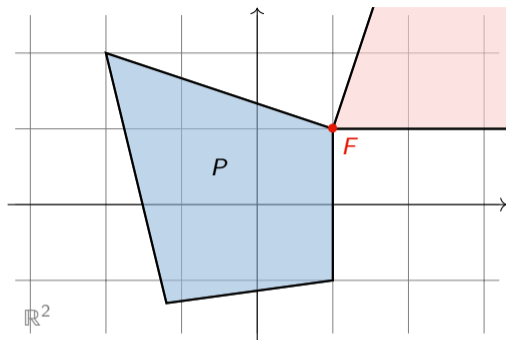
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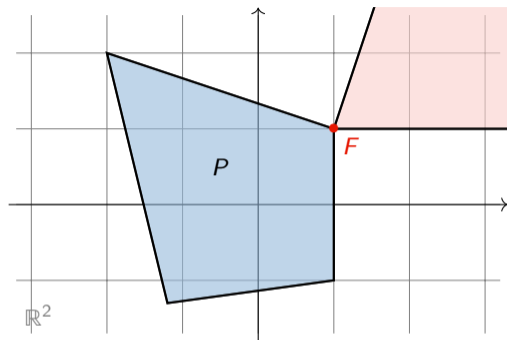
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Note: we can suppose c **integer**

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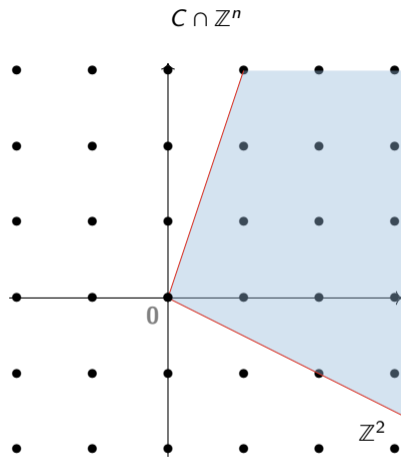
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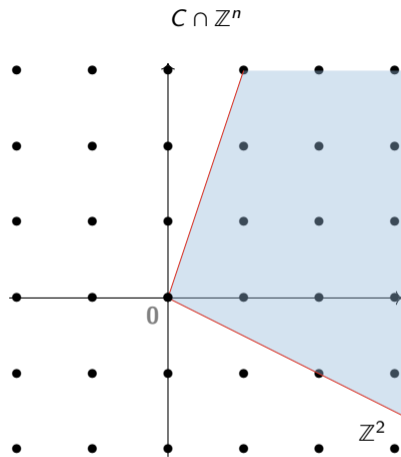
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We will study each normal cone **individually.**

TDI-ness \rightarrow generate integer vectors of the normal cones

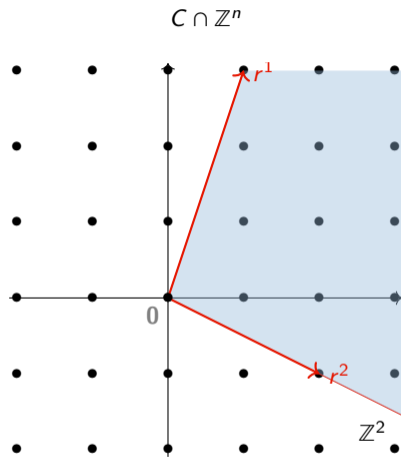


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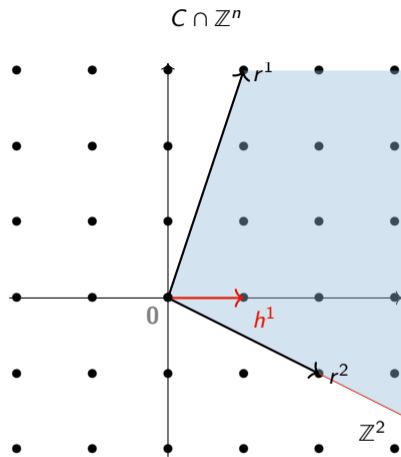
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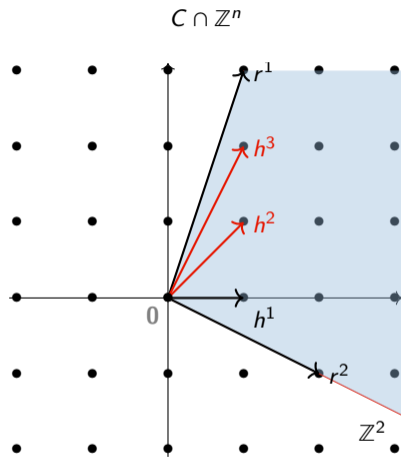
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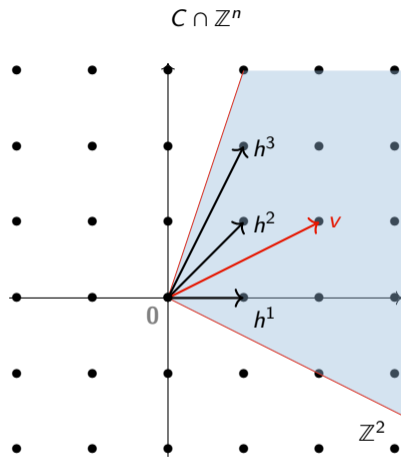
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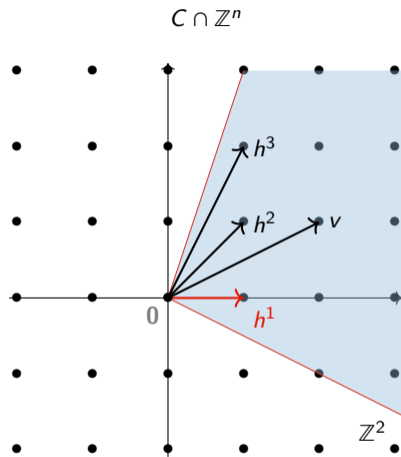
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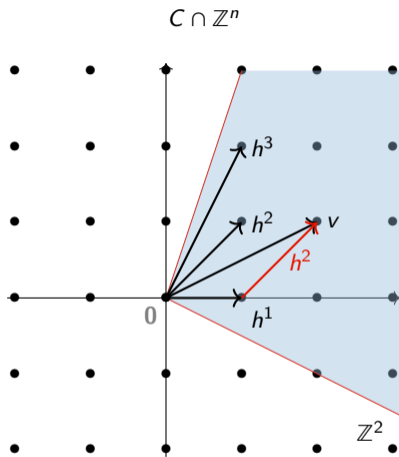
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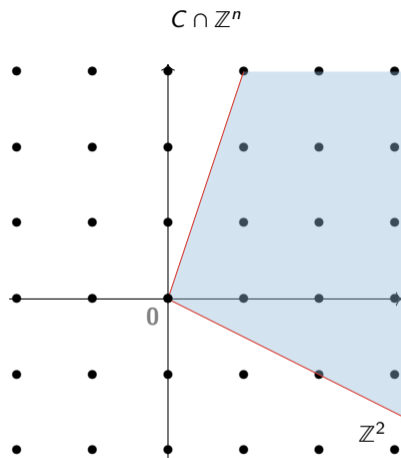
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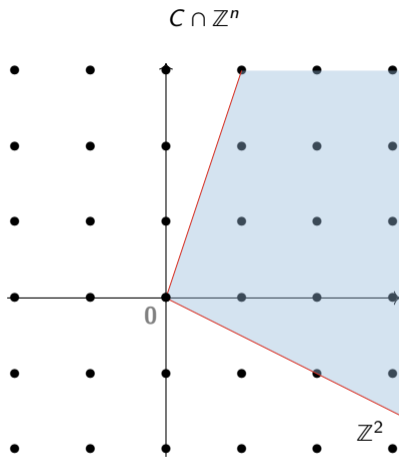
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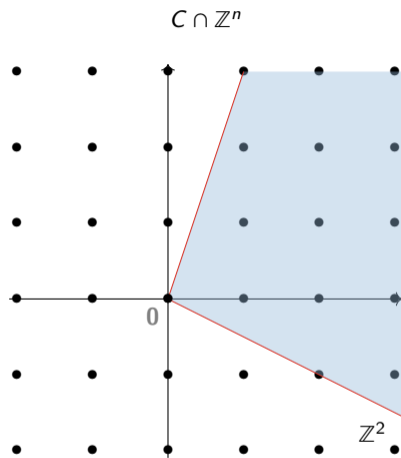
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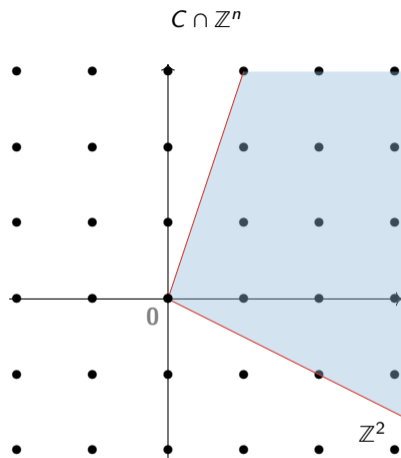
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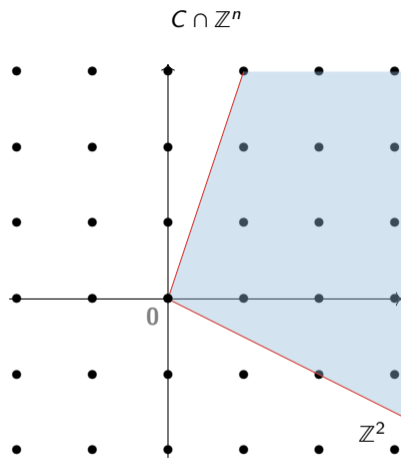
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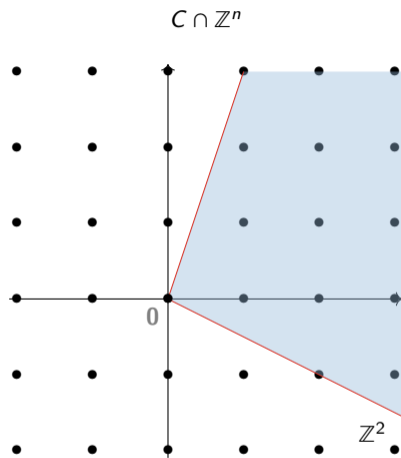
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Fact: Not every polyhedron can be described by a box-TDI system. If it can: **Box-TDI polyhedron**.

Section 2

Totally equimodular matrices: decomposition

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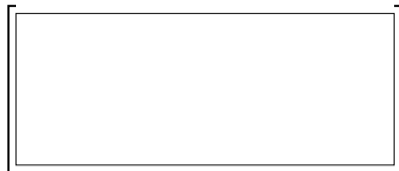
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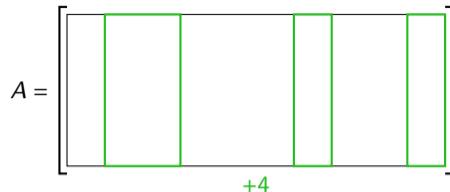
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Let $A \in \mathbb{Z}^{r \times n}$ be a matrix.

Totally unimodular (TU) matrix:

- All the nonzero subdeterminants of A are ± 1 .

Theorem (Hoffman, Kruskal, 1956)

A TU $\Leftrightarrow \forall b \in \mathbb{Z}^m, Ax \leq b$ is box-TDI.

\rightarrow incidence matrix of a network is TU \Rightarrow MaxFlow-MinCut.

Equimodular matrix:

- A has full row rank r ,
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Totally equimodular (TE) matrix:

- All the full row rank submatrices of A are equimodular.

$$A = \left[\begin{array}{c} \text{Full row rank} \Rightarrow \text{Equimodular} \end{array} \right]$$

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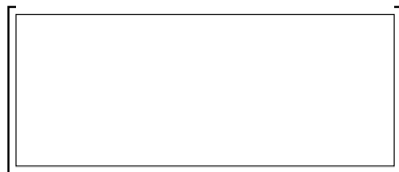
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rows of a matrix \simeq set of vectors.

Question

Is there a decomposition theorem for

TE matrices \simeq te-sets ?

\rightarrow case of **linearly independent** te-sets.

Building blocks of linearly independent te-sets with r rows

Type of te-sets

det

examples

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1

tu-set

totally unimodular

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Proposition (Chervet, Grappe, V., 2026)

The columns of a te-lace or of a thick te-interlace of size r are $\mathbf{0}$ or \pm copies of any of its nonsingular $r \times r$ submatrix.

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$$A = \underbrace{U}_{\text{tu-set}} \sqcup \underbrace{L_1 \sqcup \dots \sqcup L_k}_{\text{te-laces}} \sqcup \underbrace{S_1 \sqcup \dots \sqcup S_\ell}_{\text{thin te-interlaces}} \sqcup \underbrace{T_1 \sqcup \dots \sqcup T_m}_{\text{thick te-interlaces}}$$

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→ Cones and their Hilbert bases?

Section 3

Totally equimodular cones: triangulation

Hilbert basis of te-cones and integer decomposition properties

te-cone: cone generated by a te-set.

Theorem (Chervet, Grappe, V., 2026)

The Hilbert basis of the cone generated by a te-brick A is:

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Theorem (Carathéodory for cones)

We need at most $n = \dim C$ generators of a rational cone C to $\mathbb{Q}_{>0}$ -generate any rational $c \in C$.

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→ ICP guarantees the existence of a **sparse solution** in the dual optimization problem (TDI).

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Do totally equimodular cones satisfy the ICP?

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→ direct integer decomposition for $x \in C \cap \mathbb{Z}^n$.

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Definition (Unimodular Hilbert triangulation)

A collection of cones Σ is a UHT of C if:

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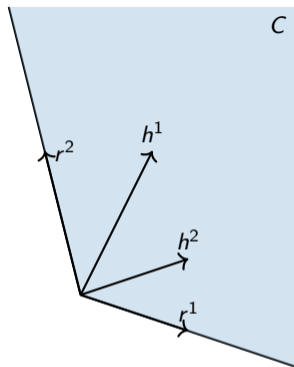
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Unimodular Hilbert triangulation

$$C = \text{cone}\{r^1, \dots, r^s\}$$

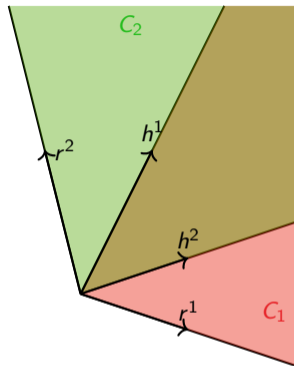
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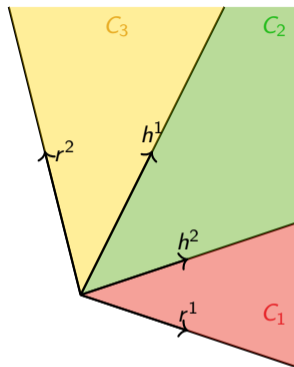
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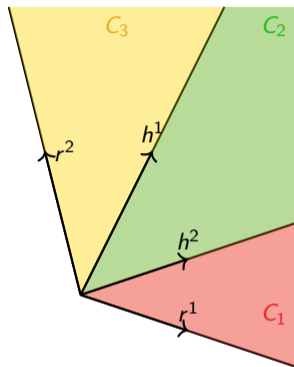
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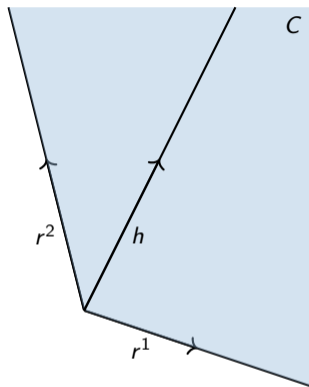
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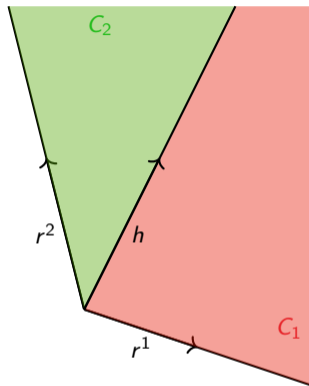
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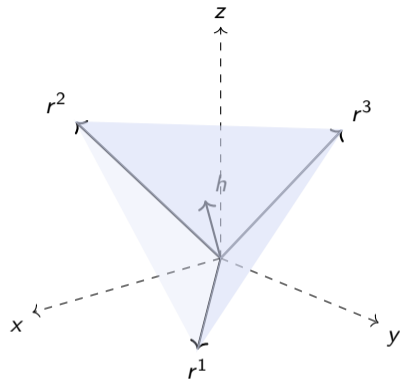
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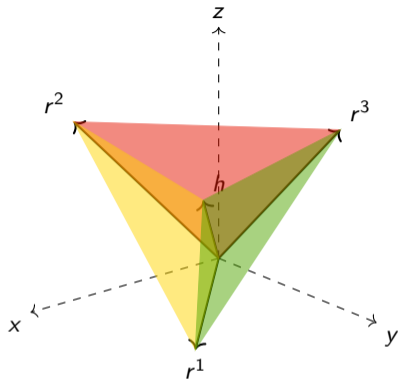
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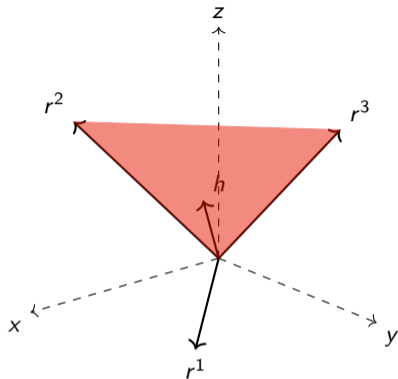
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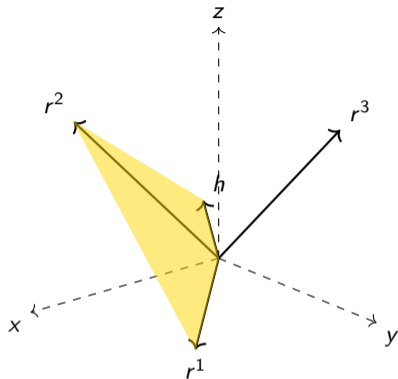
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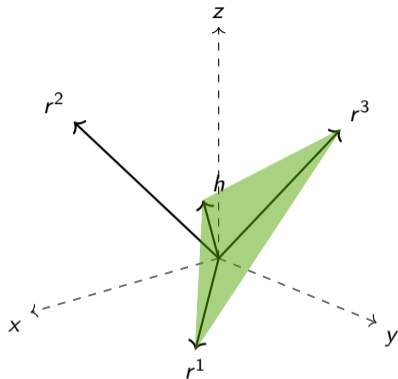
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Let C be a simplicial cone with determinant ± 2 whose all proper faces have determinant ± 1 . Then, $\mathcal{S}_h(C)$ is a RUHT, for h the half sum of its generators.

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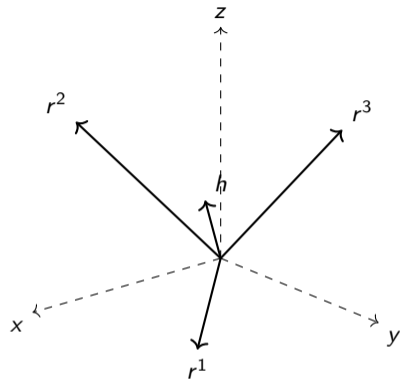
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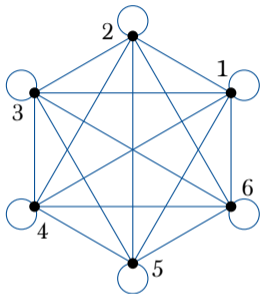
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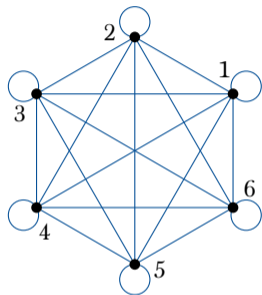
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We embed \hat{K}_n , the **complete graph** on n vertices with loops ii , as a convex n -gon in \mathbb{R}^2 :



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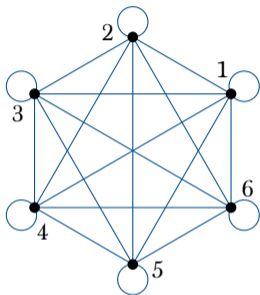
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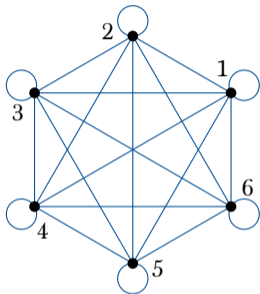
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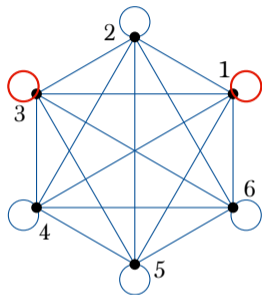


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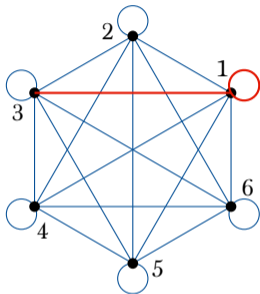


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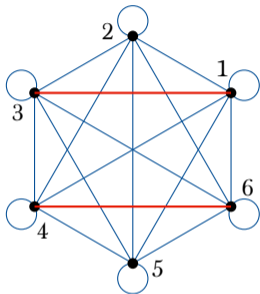


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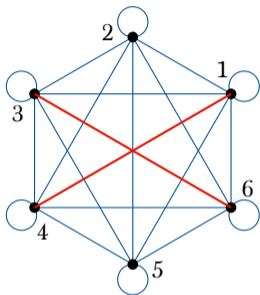


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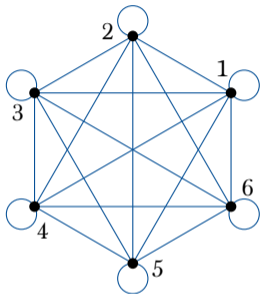
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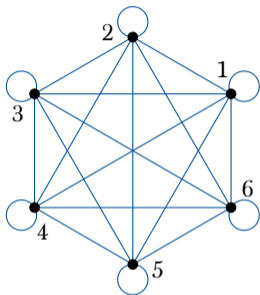


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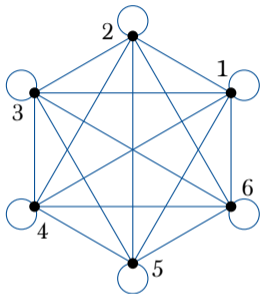
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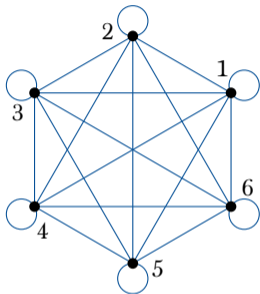
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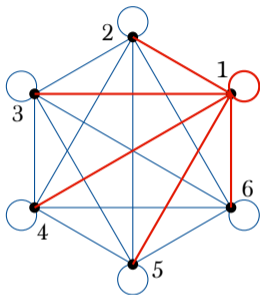
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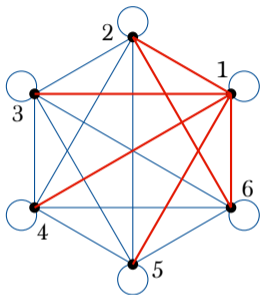
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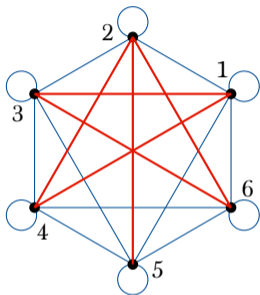
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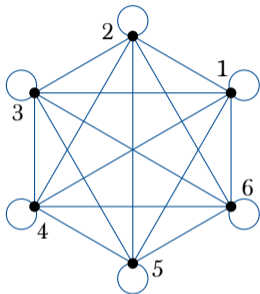
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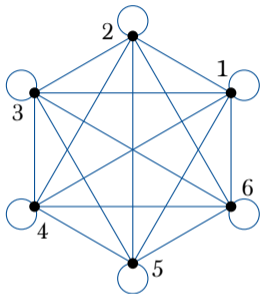
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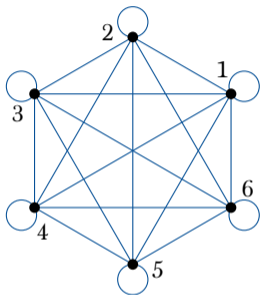
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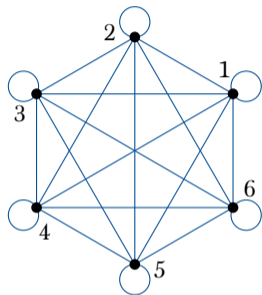
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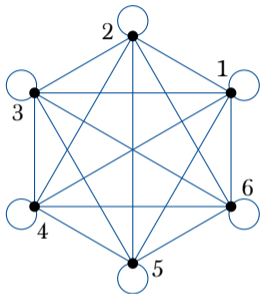
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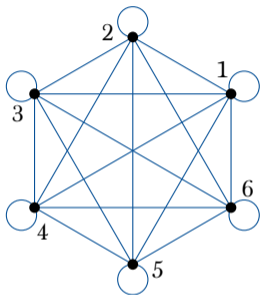
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Regular Hilbert triangulation of te-cones and a conjecture

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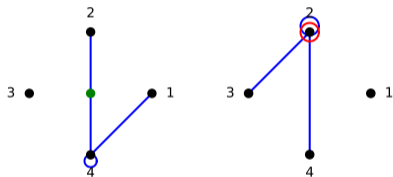
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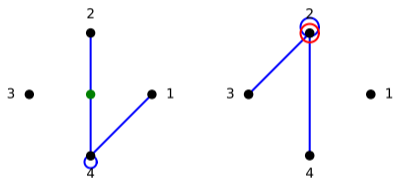


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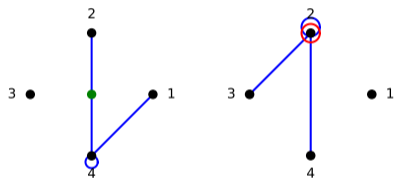
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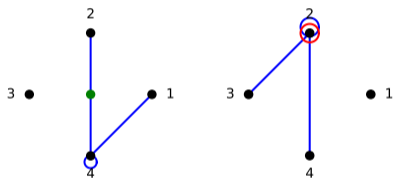
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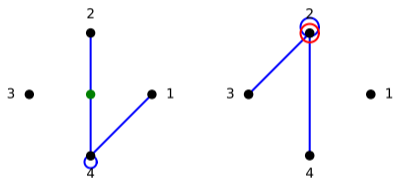
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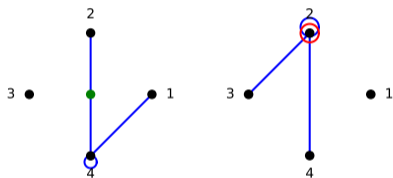
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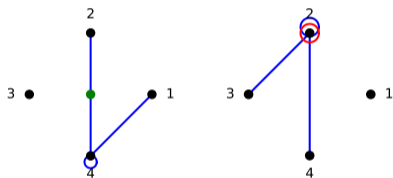
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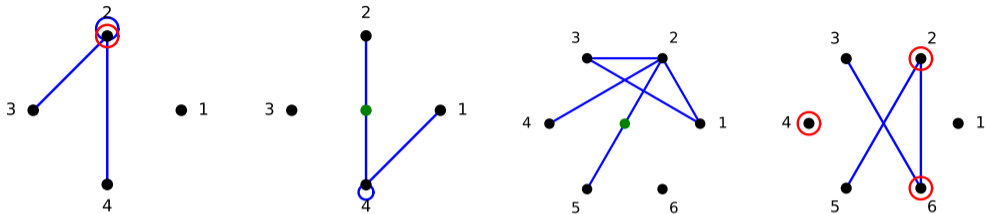
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Conjecture (Chervet, Grappe, V., 2026)

Up to resigning, there are only two thick te-interlaces:


$$(r = 4): \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}, \text{ and}$$

$$(r = 6): \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 \end{bmatrix}.$$



Thank you for listening!

Recap of the results

Type of te-sets	det	examples	Hilbert basis of cone(A)	unim. Hilbert triang.
tu-set totally unimodular	1	$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$	A	trivial: $\{\text{cone}(A)\}$
te-lace not a tu-set, every proper subset is a tu-set	2	$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$	$A \cup \{h = \frac{1}{2} \sum_i a_i\}$	stellar triangulation: $\{\text{cone}(\{h\} \cup \{a_j\}_{j \neq i})\}_i$ 
te-interlace not a tu-set, every pair forms a te-lace	thin: 2^{r-1}	$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & 0 & -1 \end{bmatrix}$	$A \cup \{\frac{1}{2}(a_i + a_j)\}_{i \neq j}$	triangulation of the second hypersimplex
	thick: 2^r	$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$	$A \cup \{\frac{1}{2}(a_i + a_j)\}_{i \neq j} \cup H$ for $H = \begin{cases} \{\frac{1}{4} \sum_i a_i\}, \text{ or} \\ \{\frac{3}{4} a_i + \frac{1}{4} \sum_{j \neq i} a_j\}_i. \end{cases}$	only for $r = 4$ and 6

On the conjecture

Let A be a 0,1 matrix and i a row index of A . The **row- i complement** $A_{[i]}$ of A is:

$$\begin{cases} A_i, & \text{if } i' = i, \\ A_{i'} + A_i \pmod{2}, & \text{otherwise.} \end{cases}$$

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Example:

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, (I_4)_{[1]} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix},$$

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Conjecture*

The only complement minimally non-TU matrices are complements of:

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$