Power set operads: the operad theory underlying polyhedral products

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Section 1

Motivation

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Definition (Polyhedral product)

K simplicial complex on $[m]$, $Y_i \subseteq X_i$:

$$\mathscr{Z}(K;(X_i,Y_i)_{i\in[m]}) = \bigcup_{I\in K}\prod_{i\in[n]}\begin{cases} X_i & i\in I, \\ Y_i & i\notin I, \end{cases}.$$

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Examples, for $(X_i, Y_i) =$

- $(D^2, S^1) \to$ Moment-angle complex.
- $(\mathbb{C}P^\infty, *) \to$ Davis–Januszkiewicz spaces.

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One can **compose** a simplicial complexes L on $[m]$ at some vertex k of another simplicial complex K on $[n]$, denoted by

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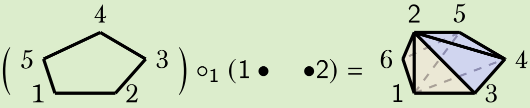

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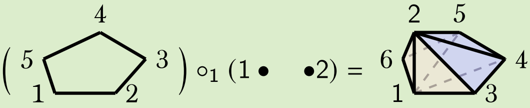

Goal

Study the interaction of these compositions with the polyhedral product through the “theory of composition of objects”= **Operad Theory**.

Some known results

Let $K(J)$ be the simplicial complex obtained after composing K with the boundary of a simplex on $[j_i]$, at every vertex i of K, for K on $[m]$ and $J = (j_1, \ldots, j_m) \in \mathbb{Z}_{>0}$.

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Theorem (Bahri-Benderski-Cohen-Gitler, 2015)

Let K be a simplicial complex on $[m]$ and $(X_i, Y_i)_{i\in[m]}$ be a sequence of pairs of topological spaces. Then for every $J = (j_1, \ldots, j_m) \in \mathbb{Z}_{>0}$, we have equality of the two following subspaces of $\prod_{i\in[m]} X_i^{j_i}$:

$$\mathfrak{X}(K(J); \underbrace{(X_1, Y_1), \ldots, (X_1, Y_1)}_{j_1}, \ldots, \underbrace{(X_m, Y_m), \ldots, (X_m, Y_m)}_{j_m})$$

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Ayzenberg, 2013: more general formula for the composition $K \circ_i L$.

Section 2

Recollections

Set theoretical settings

Definition ((Reduced) power of a set)

Let S be a set.

- Power set of S: $\wp(S) := \{X \subseteq S\}$
- Reduced power set of S: $\overline{\wp}(S) := \{\emptyset \subsetneq X \subseteq S\}$
- Set based on S with power k: element of $\wp^k(S)$.

Examples:
$\wp([2]) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$.
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Definition (Complement operations)

- $\complement X \coloneqq S \setminus X$ for $X \in \wp(S)$
- $\complement^{(1)} \coloneqq \complement$
- $\complement^{(\ell)}(K) \coloneqq \left\{\complement^{(\ell-1)} I : I \in K\right\}$ for $\ell > 1$
- Shorthand: $\complement' \coloneqq \complement^{(2)}$
- $\complement^{(\ell)} : \wp^k(S) \longrightarrow \wp^k(S)$ is an involution for $1 \leq \ell \leq k$

Examples:

$\complement\{\emptyset, \{1\}, \{2\}, \{1,2\}\} = \emptyset$.

$\complement'\{\emptyset, \{1\}, \{2\}, \{1,2\}\} = \{\{1,2\}, \{2\}, \{1\}, \emptyset\}$.

Four Families of Subsets of $[n]$

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Hypergraph H	none
Simplicial complex K	downward closed: $I \in K,\ J \subseteq I \Rightarrow J \in K$
Upward complex U	upward closed: $I \in U,\ I \subseteq J \Rightarrow J \in U$
Transversal family T	antichain: $I, J \in T,\ I \neq J \Rightarrow I \nsubseteq J$

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Categories

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Example:
$K = \{\emptyset, \{1\}, \{2\}\}$.
$U = \{\{1, 2\}\}$.
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Bridges Between Families

Key maps

- **Facets** of K: $\widehat{K}$ = incl.-max. faces $\in \mathbf{Transv}(n)$
- **Minimals** of U: $\widecheck{U}$ = incl.-min. faces $\in \mathbf{Transv}(n)$
- **Minimal non-faces**: $\mathsf{MNF}(K) = (\widecheck{\complement K}) \in \mathbf{Transv}(n)$
- **Complement involution**: $\complement K = \wp([n]) \setminus K \in \mathbf{Ucomp}(n)$

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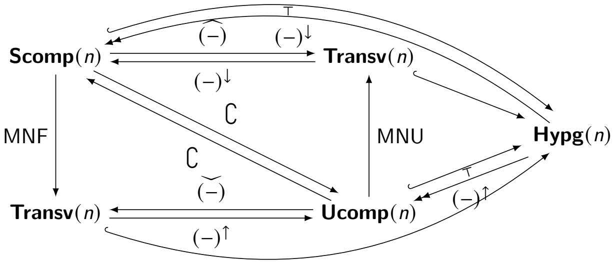

$$\begin{tikzcd}
\mathbf{Scomp}(n) \arrow[r, shift left, "\widehat{(-)}"] \arrow[dd, "\mathsf{MNF}"'] \arrow[ddr, shift left, "\complement"] \arrow[drr, hook, bend left=40, "\top"] & \mathbf{Transv}(n) \arrow[l, shift left, "(-)^{\downarrow}"] \arrow[dr, hook] & \\
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Definition (Face poset)

Let K be a (reduced) simplicial complex. The *face poset* $\mathsf{cat}(K)$ is:

- **Objects**: faces of K and $\emptyset$,
- **Morphisms**: unique arrow $J \longrightarrow I$ iff $J \subseteq I$,
- **Composition**: transitivity of $\subseteq$.

The *opposite face category* $\mathsf{cat}(K)^{\mathrm{op}}$ has arrows reversed.

Remark

The empty set $\emptyset$ is the **initial object** of $\mathsf{cat}(K)$ and the **terminal object** of $\mathsf{cat}(K)^{\mathrm{op}}$.

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A *face diagram* over K with values in C is a functor $F\colon \mathsf{cat}(K) \to \mathrm{C}$.

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$$\begin{tikzcd}
 & \{1,2\} & \\
\{1\} \arrow[ur] & & \{2\} \arrow[ul] \\
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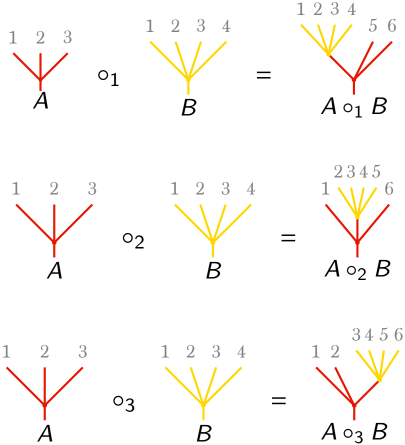
$$A\colon X^3 \longrightarrow X \quad \text{and} \quad B\colon X^4 \longrightarrow X.$$



$\mathbb{S}$-Collections and Set-Theoretical Operads

Definition ($\mathbb{S}$-collection)

A collection $\mathcal{O} = \{\mathcal{O}(n)\}_{n\in\mathbb{N}}$ of sets, each equipped with a right action $\mathcal{O}(n) \curvearrowleft \mathbb{S}_n$.

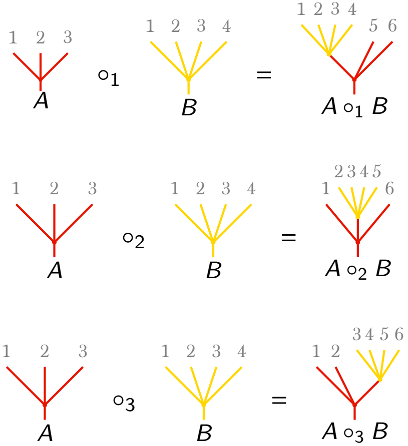
Definition (Set-theoretical operad)

An $\mathbb{S}$-collection $\mathcal{O}$ together with **composition maps**, for $n \geq 1$, $m \in \mathbb{N}$, $i = 1, \ldots, n$:

$$\circ_i \colon \mathcal{O}(n) \times \mathcal{O}(m) \longrightarrow \mathcal{O}(n+m-1)$$

compatible with the $\mathbb{S}_n$- and $\mathbb{S}_m$-actions, satisfying the two axioms on the right.

$$A \colon X^3 \longrightarrow X \quad \text{and} \quad B \colon X^4 \longrightarrow X.$$

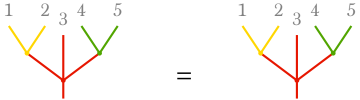


Parallel and sequential axioms

Parallel axiom

For $1 \leq i < j \leq \ell$,
$(A, B, C) \in \mathcal{O}(\ell) \times \mathcal{O}(n) \times \mathcal{O}(m)$:

$$(A \circ_i B) \circ_{j+m-1} C = (A \circ_j C) \circ_i B$$



$$(A \circ_1 B) \circ_4 C \quad = \quad (A \circ_3 C) \circ_1 B$$

B and C are at *independent* branches of A order of grafting does not matter.

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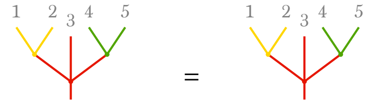

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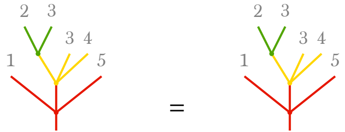
Sequential axiom

For $1 \leq i \leq \ell$, $1 \leq j \leq m$:

$$A \circ_i (B \circ_j C) = (A \circ_i B) \circ_{j+i-1} C$$



$$A \circ_2 (B \circ_1 C) \qquad (A \circ_2 B) \circ_2 C$$

C nested inside B inside A both sides yield the same tree.

Free Operad and rewriting

Free operad $\mathcal{T}M$

Given an $\mathbb{S}$-collection M, the *free operad* $\mathcal{T}M$ is the left adjoint to the forgetful functor $\mathsf{Op} \to \mathbb{S}$-collections.

Its operations are rooted trees with internal nodes decorated by M:

- atomic corollas with n leaves decorated by $m \in M(n)$,
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Relations and rewriting

A **relation** $T_1 \equiv T_2$ identifies two trees of the same arity decorated by M.

Given $T \in \mathcal{T}M$ and a relation $T_1 \equiv T_2$:

$$T_1 \subseteq T \implies T_1 \to T_2 \text{ in } T$$

(replace the subtree T_1 by T_2).

Two trees $T \sim T'$ iff there is a sequence of such rewritings.

Free Operad and rewriting

Free operad $\mathcal{T} M$

Given an $\mathbb{S}$-collection M, the *free operad* $\mathcal{T} M$ is the left adjoint to the forgetful functor $\mathsf{Op} \to \mathbb{S}$-collections.

Its operations are rooted trees with internal nodes decorated by M:

- atomic corollas with n leaves decorated by $m \in M(n)$,
- finite trees built by grafting, up to $\mathbb{S}_n$-action.

Relations and rewriting

A **relation** $T_1 \equiv T_2$ identifies two trees of the same arity decorated by M.

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(replace the subtree T_1 by T_2).

Two trees $T \sim T'$ iff there is a sequence of such rewritings.

Definition (Presentation)

An operad $\mathcal{O}$ *admits the presentation* (M, R) if

$$\mathcal{O} \cong \mathcal{T} M / \langle R \rangle.$$

Section 3

Examples of (set) operads and their algebras

Example: *Perm*

Definition (Permutative operad, Chapoton 2001)

The operad *Perm* has:

$Perm(n) := [n]$ for $n \geq 1$, $\quad Perm(0) = \emptyset$.

$\mathbb{S}_n$-action: natural action on $[n]$.

Composition: for $(i, j) \in [n] \times [m]$ and $k = 1, \ldots, n$,

$$i \circ_k j = \begin{cases} i + m - 1 & k < i, \\ i + j - 1 & k = i, \\ i & k > i. \end{cases}$$

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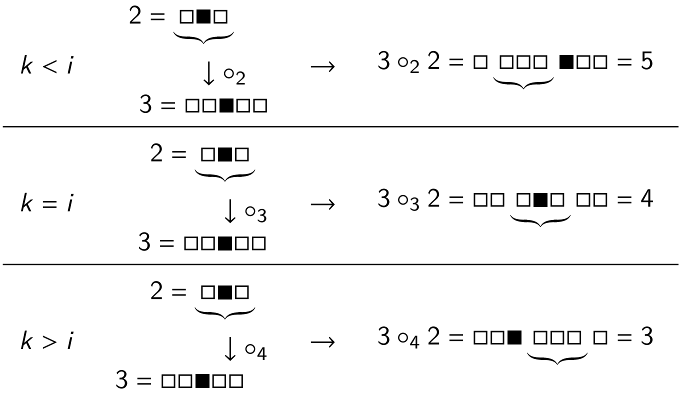


$k < i$: $2 = \underbrace{\square\blacksquare\square}$, $\downarrow \circ_2$, $3 = \square\square\blacksquare\square\square$ $\longrightarrow$ $3 \circ_2 2 = \square\ \underbrace{\square\square\square}\ \blacksquare\square\square = 5$

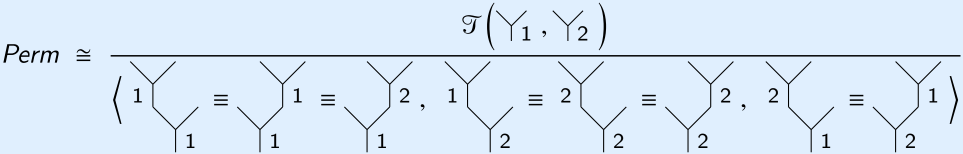
$k = i$: $2 = \underbrace{\square\blacksquare\square}$, $\downarrow \circ_3$, $3 = \square\square\blacksquare\square\square$ $\longrightarrow$ $3 \circ_3 2 = \square\square\ \underbrace{\square\blacksquare\square}\ \square\square = 4$

$k > i$: $2 = \underbrace{\square\blacksquare\square}$, $\downarrow \circ_4$, $3 = \square\square\blacksquare\square\square$ $\longrightarrow$ $3 \circ_4 2 = \square\square\blacksquare\ \underbrace{\square\square\square}\ \square = 3$

Figure: An illustration of the compositions of two elements of *Perm*. The element $i \in Perm(n)$ is represented by a sequence of n squares, where only the ith one is colored in black.

Presentation of *Perm*

Presentation, Chapoton 2001



The transposition (12) sends $\curlyvee_1$ to $\curlyvee_2$.

Algebra over an Operad

Endomorphism operad

For any set X, the operad End_X is defined by:

$$End_X(n) := \mathrm{Hom}_{\mathsf{Set}}(X^n, X),$$

with $\mathbb{S}_n$ acting by permuting entries, and:

$$\begin{aligned} f \circ_i g\,(x_1, \dots, \widehat{x}_i, \dots, x_n;\, y_1, \dots, y_m) \\ = f(x_1, \dots, g(y_1, \dots, y_m), \dots, x_n). \end{aligned}$$

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Definition ($\mathcal{O}$-algebra)

Given an operad $\mathcal{O}$, an $\mathcal{O}$*-algebra* structure on X is a morphism of operads

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Equivalent description

A sequence of maps $\varkappa = (\varkappa_n)_{n \geq 1}$:

$$\begin{aligned} \varkappa_n : \quad \mathcal{O}(n) \times X^n \quad & \longrightarrow X \\ (A;\ x_1, \dots, x_n) \quad & \longmapsto \varkappa_n(A;\ x_1, \dots, x_n) \end{aligned}$$

compatible with:

- the right $\mathbb{S}_n$-action on $\mathcal{O}(n)$,
- the left $\mathbb{S}_n$-action on X^n (perm. entries).

and, the following diagram commutes:

$$\begin{tikzcd} \mathcal{O}(n) \times \mathcal{O}(m) \times X^{n+m-1} \arrow[r, "\mathrm{id} \times \varkappa_m"] \arrow[d, "\circ_i \times \mathrm{id}"] & \mathcal{O}(m) \times X^m \arrow[d, "\varkappa_m"] \\ \mathcal{O}(n+m-1) \times X^{n+m-1} \arrow[r, "\varkappa_{n+m-1}"] & X \end{tikzcd},$$

Algebra over an Operad: Examples

Key examples:

- Perm-algebras = *permutative algebras*
- Com-algebras = *commutative algebras*

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The *commutative operad* Com has **one operation per arity**:

$$\mathit{Com}(n) := \{\mu_n\}, \quad n \geq 1.$$

The $\mathbb{S}_n$-action is trivial.

Presentation with a single binary generator $\mu := \mu_2$:

$$\mathit{Com} \cong \frac{\mathcal{T}(\mu)}{\langle \mu \circ_1 \mu \equiv \mu \circ_2 \mu \rangle}.$$

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Com-algebras = commutative algebras

An *Com*-algebra structure on X is given by maps

$$\mathscr{z}_n \colon \{\mu_n\} \times X^n \to X,$$

i.e. a single n-ary operation on X for each n.

The binary generator μ gives:

$$\begin{aligned} \cdot \, : X \times X &\to X, \\ (x, y) &\mapsto x \cdot y := \mathscr{z}_2(\mu; x, y). \end{aligned}$$

The relation $\mu \circ_1 \mu \equiv \mu \circ_2 \mu$ becomes:

$$(x \cdot y) \cdot z = x \cdot (y \cdot z).$$

All higher μ_n are then determined by μ via composition.

Section 4

Power set operad

Power Set Functor: Lax Monoidal Structure

Lemma

The power set functor $\wp$ is a **lax monoidal** endofunctor of $(\mathsf{Set}, \times, \{*\})$:

$$\begin{array}{rcl} \{*\} & \longrightarrow & \wp(\{*\}) \\ * & \longmapsto & \{*\} \end{array}$$

$$\begin{array}{rcl} \wp(X) \times \wp(Y) & \longrightarrow & \wp(X \times Y) \\ (I, J) & \longmapsto & I \times J \end{array}$$

The **reduced** power set functor $\overline{\wp}$ is lax monoidal with the same formula, landing in $\overline{\wp}(X \times Y)$.

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Definition (Power of a collection)

Given an $\mathbb{S}$-collection $\mathbb{O}$, its *power collection* is:

$$\wp\mathbb{O} := \{\wp(\mathbb{O}(n))\}_{n \in \mathbb{N}},$$

with the induced $\mathbb{S}_n$-action.

The *reduced* power collection $\overline{\wp}\mathbb{O}$ is defined analogously.

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Definition (Iterated power)

A collection $\mathbb{Q}$ is *based on* $\mathbb{O}$ *with power* $k \geq 0$ if $\mathbb{Q} \subseteq \wp^k\mathbb{O}$,

where $\wp^0\mathbb{O} := \mathbb{O}$.

Power Set Operad Functor

Definition (Power set operad functor)

The *power set operad functor* is:

$$\begin{aligned} \wp \colon \mathsf{Op}(\mathsf{Set}) &\longrightarrow \mathsf{Op}(\mathsf{Set}), \\ (\mathcal{O}, \{\circ_i\}) &\mapsto (\wp\mathcal{O}, \{\circ_i^{\wp}\}), \end{aligned}$$

with compositions:

$$\begin{aligned} \circ_i^{\wp} \colon \wp\mathcal{O}(n) \times \wp\mathcal{O}(m) &\longrightarrow \wp\mathcal{O}(n+m-1) \\ (A, B) &\mapsto \{a \circ_i b : a \in A,\ b \in B\}. \end{aligned}$$

We call $(\wp\mathcal{O}, \{\circ_i^{\wp}\})$ the **power** of $(\mathcal{O}, \{\circ_i\})$.

The *reduced* variant $\overline{\wp}$ gives the **reduced power operad** $(\overline{\wp}\mathcal{O}, \{\circ_i^{\overline{\wp}}\})$.

Proof: Either check by hand of use lax monoidality. □

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If $A = \emptyset$ or $B = \emptyset$, then $A \circ_i^{\wp} B = \emptyset$.

This is why the *reduced* variant $\overline{\wp}$ (excluding $\emptyset$) is often more natural.

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Complement composition

The bijection $\complement$ on $\wp\mathcal{O}$ allows transporting $\{\circ_i^{\wp}\}$ to a new composition:

$$\circ_i^{c\wp} \colon \wp\mathcal{O}(n) \times \wp\mathcal{O}(m) \to \wp\mathcal{O}(n+m-1),$$
$$(A, B) \mapsto \complement\{a \circ_i b : a \in \complement A,\ b \in \complement B\}.$$

This yields a functor **isomorphic to** $\wp$ via $\complement$.

Structure of the Power Operad

Proposition (V., 2026+)

Let $\mathcal{O}$ be a set operad with $\mathcal{O}(0) = \emptyset$.

1. *The collection $\{\emptyset_n\}_{n\geq 1}$ is a* ***non-unital suboperad*** *of $\wp\mathcal{O}$, isomorphic to Com (non-unital), and an* ***operadic ideal****.*
2. *The inclusion $\overline{\wp}\mathcal{O} \hookrightarrow \wp\mathcal{O}$ is unital and provides a* ***canonical splitting****.*

In particular, there is a ***split short exact sequence*** *of non-unital operads:*

$$0 \longrightarrow Com \longrightarrow \wp\mathcal{O} \longrightarrow \overline{\wp}\mathcal{O} \longrightarrow 0,$$

yielding an arity-wise decomposition:

$$\wp\mathcal{O}(n) = \overline{\wp}\mathcal{O}(n) \sqcup \{\emptyset_n\}.$$

→ We can treat the empty sets separately.

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Examples of $\overline{\wp}\mathcal{O}$

Trivial operad Triv

Triv has a single operation id of arity 1.

$$\overline{\wp}\,\mathit{Triv}(1) = \{\{\mathrm{id}\}\}, \quad \{\mathrm{id}\} \circ_1 \{\mathrm{id}\} = \{\mathrm{id}\}.$$

$$\overline{\wp}\,\mathit{Triv} \cong \mathit{Triv}\,.$$

Commutative operad Com

Com has a single operation μ_n of arity n for every $n \geq 1$, with trivial $\mathbb{S}_n$-action.

$$\overline{\wp}\,\mathit{Com}(n) = \{\{\mu_n\}\} \quad \text{for all } n \geq 1.$$

$$\overline{\wp}\,\mathit{Com} \cong \mathit{Com}\,.$$

Both Triv and Com are **fixed points** of $\overline{\wp}$.

The Reduced Power of *Perm*

We have $\overline{\wp}(Perm(n)) = \overline{\wp}([n])$ for $n \geq 1$.

For $(I, J) \in \overline{\wp}([n]) \times \overline{\wp}([m])$ and $k \in [n]$:

$$I \circ_k J = \{i \circ_k j : i \in I, j \in J\}$$

Splitting by cases on i vs k:

$$I \circ_k J = \begin{cases} I^{<k} \sqcup (J + k - 1) \sqcup (I^{>k} + m - 1) & k \in I, \\ I^{<k} \sqcup (I^{>k} + m - 1) & k \notin I, \end{cases}$$

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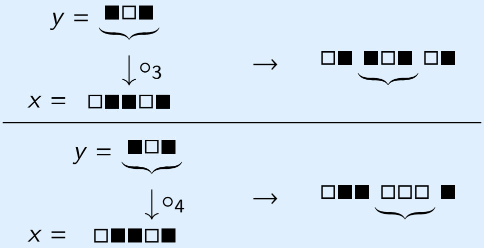

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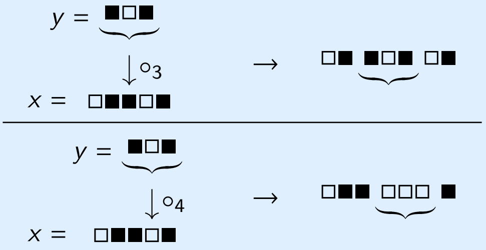

Proposition (V., 2026+)

We have

$$\overline{\wp}(Perm) \cong ComTrias,$$

where $ComTrias$ *is some operad introduced by Vallette in 2007.*

An algebra over $\overline{\wp}(\mathit{Perm})$

Proposition (V., 2026+)

Let $(C, \otimes, e)$ *be a* *monoidal category. Then, the set* $\mathrm{mor}(\mathrm{C})$ *of all arrows of* C *carries* $\overline{\wp}(\mathit{Perm})$*-algebra structure given by:*

$$\varkappa_n : \overline{\wp}(\mathit{Perm})(n) \times \mathrm{mor}(\mathrm{C})^n \longrightarrow \mathrm{mor}(\mathrm{C})$$

where

$$\varkappa_n(I; Y_1 \xrightarrow{f_1} X_1, \ldots, Y_n \xrightarrow{f_n} X_n) := \otimes_{i=1}^n g_i \; : \; \bigotimes_{i=1}^n Y_i \longrightarrow \bigotimes_{i=1}^n \begin{cases} X_i & i \in I \\ Y_i & i \notin I \end{cases},$$

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An algebra over $\overline{\wp}(Perm)$

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Iterating $\overline{\wp}$ on *Perm*: the reduced substitution operad

We have:

$$\overline{\wp}^2(\mathit{Perm})(n) = \overline{\wp}\big(\overline{\wp}([n])\big) = \overline{\mathsf{hypg}}(n).$$

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Definition (Reduced substitution operad)

Denoted by $(\overline{\mathrm{hypg}}, \{\circ_k\})$.

For $K \in \overline{\mathrm{hypg}}(n)$, $L \in \overline{\mathrm{hypg}}(m)$, $k \in [n]$:

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→ We recover the **substitution of L into K at vertex** k in the sense of Abramyan–Panov (2019), in the case of reduced hypergraphs.

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Proposition (V., 2026+)

The pairs $(\overline{\mathsf{transv}}, \{\circ_k\})$ *and* $(\overline{\mathsf{scpx}}, \{\circ_k\})$ *form* ***suboperads*** *of* $(\overline{\mathsf{hypg}}, \{\circ_k\})$.

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Definition (Reduced substitution operad)

Denoted by $(\overline{\mathsf{hypg}}, \{\circ_k\})$.

For $K \in \overline{\mathsf{hypg}}(n)$, $L \in \overline{\mathsf{hypg}}(m)$, $k \in [n]$:

$$\begin{aligned} K \circ_k L \; = & \left\{ I^{<k} \sqcup (J + k - 1) \sqcup (I^{>k} + m - 1) \right. \\ & \qquad \left. : I \in K,\ J \in L,\ k \in I \right\} \\ & \cup \left\{ I^{<k} \sqcup (I^{>k} + m - 1) : I \in K,\ k \notin I \right\}. \end{aligned}$$

→ We recover the **substitution of L into K at vertex k** in the sense of Abramyan–Panov (2019), in the case of reduced hypergraphs.

Proposition (V., 2026+)

The pairs $(\overline{\mathsf{transv}}, \{\circ_k\})$ *and* $(\overline{\mathsf{scpx}}, \{\circ_k\})$ *form* ***suboperads of*** $(\overline{\mathsf{hypg}}, \{\circ_k\})$.

Summary: iterating $\overline{\wp}$ on *Perm*

$$\begin{tikzcd} \mathit{Perm}(n) = [n] \arrow[d, "\overline{\wp}"] \\ \overline{\wp}\,\mathit{Perm}(n) = \overline{\wp}([n]) \arrow[d, "\overline{\wp}"] \\ \overline{\wp}^2\,\mathit{Perm}(n) = \overline{\mathsf{hypg}}(n) \end{tikzcd}$$

Suboperads at level 2:

$$\overline{\mathsf{transv}}(n),\ \overline{\mathsf{scpx}}(n) \hookrightarrow \overline{\mathsf{hypg}}(n).$$

From *Perm* to Substitution and Composition Operads

The operad *IdemCom*

Extend $\overline{\wp}(\mathit{Perm})$ by allowing $\emptyset$:

$$\mathit{IdemCom}(n) := \wp([n]), \quad n \geq 1.$$

$\Rightarrow \wp(\mathit{IdemCom})(n) = \mathrm{hypg}(n)$.

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Suboperads of $(\mathrm{hypg}, \{\circ_k\})$

- $(\mathrm{scpx}, \{\circ_k\})$, $(\mathrm{scpx}_{\neq\emptyset}, \{\circ_k\})$: suboperads.
- transv, $\overline{\mathrm{transv}}$: **not** stable.

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Composition operation on *IdemCom*

The complement $\complement : \wp([n]) \xrightarrow{\sim} \wp([n])$ transports $\{\circ_k\}$ to a new composition on *IdemCom*:

$$I \circ_k^c J := \complement\left(\complement(I) \circ_k \complement(J)\right),$$

i.e. explicitly:

$$I \circ_k^c J = \begin{cases} I^{<k} \sqcup ([m] + k{-}1) \sqcup (I^{>k} + m{-}1) & k \in I, \\ I^{<k} \sqcup (J + k{-}1) \sqcup (I^{>k} + m{-}1) & k \notin I. \end{cases}$$

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Proposition

- $(\mathrm{ucpx}, \{\circ_i^c\}) \hookrightarrow (\mathrm{hypg}, \{\circ_i^c\})$: suboperad.
- $(\mathrm{scpx}, \{\circ_i^c\})$: composition operad, recovering Ayzenberg's composition after $(-)^{\downarrow}$.

Section 5

Algebras over the operads on simplicial complexes

Polyhedral Product

Definition (Polyhedral product)

Let $(\underline{X}, \underline{Y}) = ((X_i, Y_i))_{i \in [n]}$ be a sequence of pairs of spaces with $Y_i \subseteq X_i$.

The *polyhedral product* is the functor

$$\mathcal{Z}_n(-;\, (\underline{X}, \underline{Y}))\colon \mathsf{scpx}_{\neq\emptyset}(n) \longrightarrow \mathsf{pairTop},$$

$$K \mapsto \left(\prod_{i \in [n]} X_i,\ \underset{I \in \mathsf{cat}(K)}{\operatorname{colim}} \, z_n(I;\, (\underline{X}, \underline{Y})) \right),$$

where the face diagram is:

$$z_n(I;\, (\underline{X}, \underline{Y})) := \prod_{i \in [n]} \begin{cases} X_i & i \in I, \\ Y_i & i \notin I, \end{cases}$$

with arrows given by natural inclusions $I \subseteq J$.

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Proposition (Ayzenberg 2013)

Let $K \in \mathsf{scpx}_{\neq \emptyset}(n)$, $L \in \mathsf{scpx}_{\neq \emptyset}(m)$, $k \in [n]$. *For every* $((X_i, Y_i))_{i \in [n+m-1]}$:

$$\mathscr{Z}^c_{n+m-1}(K \circ^c_k L;\, (\underline{X}, \underline{Y})) = \mathscr{Z}^c_n(K;\, (\underline{X}', \underline{Y}')),$$

where for $i \in [n]$:

$$(X'_i, Y'_i) := \begin{cases} (X_i, Y_i) & i < k, \\ \mathscr{Z}^c_m(L;\, (X_j, Y_j)_{k \leq j < k+m}) & i = k, \\ (X_{i+m-1}, Y_{i+m-1}) & i > k. \end{cases}$$

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$$\mathcal{Z}^c_{n+m-1}(K \circ^c_k L;\,(\underline{X},\underline{Y})) = \mathcal{Z}^c_n(K;\,(\underline{X'},\underline{Y'})),$$

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Operadic rephrasing

$\mathsf{pairTop}$ carries a $(\mathsf{scpx}_{\neq\emptyset}, \{\circ^c_i\})$-**algebra** structure, with structure map the polyhedral product functor.

Algebra over the Substitution Operad

Theorem (V., 2026+)

Let $(\mathrm{C}, \otimes, e)$ be a small strict cocontinuous cocomplete symmetric monoidal category.

Then $\mathsf{mor}(\mathrm{C})$ carries a $(\mathsf{scpx}_{\neq\emptyset}, \{\circ_i\})$-**algebra** structure, with structure map:

$$\mathscr{X}_n \colon \mathsf{scpx}_{\neq\emptyset}(n) \times \mathsf{mor}(\mathrm{C})^n \longrightarrow \mathsf{mor}(\mathrm{C}),$$

$$\mathscr{X}_n(K;\, f_1, \ldots, f_n) = \Big(Y_1 \otimes \cdots \otimes Y_n \longrightarrow \operatorname*{colim}_{I \in \mathsf{cat}(K)} \varkappa_n(I;\, (\underline{X}, \underline{Y})) \Big),$$

where $f_i \colon Y_i \longrightarrow X_i$ and

$$\varkappa_n(I;\, (\underline{X}, \underline{Y})) := \bigotimes_{i=1}^{n} \begin{cases} X_i & i \in I, \\ Y_i & i \notin I. \end{cases}$$

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For the composition operad

For, $(\mathsf{scpx}_{\neq\emptyset}, \{\circ_i^c\})$:

$$\mathscr{X}_n^c(K; (\underline{X}, \underline{Y}))$$

$$= \Big(\operatorname*{colim}_{I \in \mathsf{cat}(K)} \mathscr{z}_n(I; (\underline{X}, \underline{Y})) \longrightarrow \bigotimes_{i=1}^{n} X_i \Big).$$

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Example: C = Top with ⊗ = ×

$$\mathrm{mor}(\mathrm{Top}) = \mathrm{pairTop},$$

$$\mathscr{Z}_n(K; -) = \text{polyhedral product.}$$

Same story for limits

Since $\mathsf{cat}(K)^{\mathrm{op}}$ reverses the face diagram, we use **limits** instead of colimits (or pass $(C, \otimes, e)$ through a "nice" contravariant functor).

Theorem (V., 2026+)

Assuming $(\mathrm{C}, \otimes, e)$ is small, strict, continuous, and complete, $\mathsf{mor}(\mathrm{C})$ carries an algebra structure:

(i) Over $(\mathsf{scpx}_{\neq\emptyset}, \{\circ_i\})$:

$$\mathscr{L}_n(K;\, (\underline{X}, \underline{Y})) := \left(\lim_{I \in \mathsf{cat}(K)^{\mathrm{op}}} \ell_n(I;\, (\underline{X}, \underline{Y})) \to \bigotimes_{i=1}^{n} X_i \right),$$

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Example: $\mathrm{C} = \mathsf{Top}$, $\otimes = \wedge$

(some weakening of the hypotheses)

$$\mathscr{L}_n(K;\, -) = \text{polyhedral coproduct.}$$

(See Amelotte-Hornsilien-Stanton, 2024)

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All four maps

Map	Direction
$\mathscr{E}_n$	$\otimes Y_i \longrightarrow \mathsf{colim}$
$\mathscr{E}_n^c$	$\mathsf{colim} \longrightarrow \otimes X_i$
$\mathscr{L}_n$	$\lim \longrightarrow \otimes X_i$
$\mathscr{L}_n^c$	$\otimes Y_i \longrightarrow \lim$

Section 6

Simplicial join operad

The Simplicial Join Operad on Pairs

Definition (Polyhedral join product)

K a simplicial complex on $[m]$ and $(M_i, N_i)_{i\in[m]}$ pairs on simplicial complexes, on $[n_i]$

$$\mathcal{Z}^*_n(K;(M_i,N_i)) = \bigcup_{I\in K} \mathop{\ast}_{i\in[m]} \begin{cases} M_i & i\in I \\ N_i & i\notin I \end{cases} \subseteq \mathop{\ast}_{i\in[m]} M_i.$$

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Definition (Partial join product)

For K on $[n]$, a pair (M, N) on $[m]$, and $k \in [n]$, the *partial polyhedral join product* is:

$$\begin{aligned} K \triangleleft^*_k (M, N) &\\ &:= \mathcal{Z}^*_n\big(K;\ (\mathrm{pt}, \{\emptyset\}), \dots, \underbrace{(M, N)}_{k}, \dots, (\mathrm{pt}, \{\emptyset\})\big) \\ &= \big(\mathrm{lk}_k(K) *_k M\big) \cup \big((K \setminus k) *_k N\big). \end{aligned}$$

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Join composition on pairs

For pairs (K, L) on $[n]$ and (M, N) on $[m]$:

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Theorem (V., 2026+)

The pair $(\mathsf{pairscpx}, \{\circ^*_i\})$ is a unital operad with unit $(\Delta_{[1]} = \{\emptyset, \{1\}\}, \{\emptyset\})$.

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Suboperads

Both previous operads are special cases:

$$(K \circ_k L, \{\emptyset\}) = (K, \{\emptyset\}) \circ_k^* (L, \{\emptyset\}),$$

$$(\Delta_{[n+m-1]}, K \circ_k^c L) = (\Delta_{[n]}, K) \circ_k^* (\Delta_{[m]}, L).$$

Algebra over the simplicial join operad

Theorem (V., 2026+)

Let $(\mathrm{C}, \otimes, e)$ be a small strict cocontinuous cocomplete symmetric monoidal category. Then, there is a $(\mathsf{pairscpx}_{\neq(-,\emptyset)}, \{\circ_i^*\})$-algebra structure on $\mathsf{mor}(\mathrm{C})$ given by

$$\begin{aligned} \mathscr{M}_n \colon \mathsf{pairscpx}_{\neq\emptyset}(n) \quad &\times \quad \mathsf{mor}(C)^n &&\longrightarrow \mathsf{mor}(C) \\ ((K, L) \quad &; \quad (\underline{X}, \underline{Y})) &&\mapsto \left(\operatorname*{colim}_{I \in \mathsf{cat}(L)} \mathscr{z}_n(I; (\underline{X}, \underline{Y})) \longrightarrow \operatorname*{colim}_{I \in \mathsf{cat}(K)} \mathscr{z}_n(I; (\underline{X}, \underline{Y})) \right). \end{aligned}$$

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What remains to do

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What remains to do

1. Study algebras on suboperads $\mathcal{S}$ of the composition and substitution operads, and $\mathbb{S}$-subcollections supporting an $\mathcal{S}$-module structure
   e.g. $\mathsf{scpx}_{\neq\emptyset}$ and right $(\{\partial\Delta^n\}, \circ_i^c)$-module $\rightarrow K(J)$.

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2. Study algebraic properties like Gorenstein(*)-ness, Golodness, minimally non-Golodness.

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$$\begin{array}{rccl} \mathscr{M}_n \colon \mathsf{pairscpx}_{\neq\emptyset}(n) & \times & \mathsf{mor}(C)^n & \longrightarrow \mathsf{mor}(C) \\ ((K, L) & ; & (\underline{X}, \underline{Y})) & \longmapsto \left(\operatorname*{colim}_{I \in \mathsf{cat}(L)} \varkappa_n(I; (\underline{X}, \underline{Y})) \longrightarrow \operatorname*{colim}_{I \in \mathsf{cat}(K)} \varkappa_n(I; (\underline{X}, \underline{Y})) \right). \end{array}$$

What remains to do

1. Study algebras on suboperads $\mathcal{S}$ of the composition and substitution operads, and $\mathbb{S}$-subcollections supporting an $\mathcal{S}$-module structure
   e.g. $\mathsf{scpx}_{\neq\emptyset}$ and right $(\{\partial\Delta^n\}, \circ_i^c)$-module $\longrightarrow K(J)$.
2. Study algebraic properties like Gorenstein(*)-ness, Golodness, minimally non-Golodness.
3. Find the Koszul dual to the composition and substitution operads over simplicial complexes.

Algebra over the simplicial join operad

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5. Does the Cartan-type formula of [BBCG, 2024] comes from an operadic construction?

Thank you for listening!