Totally equimodular matrices : decomposition and triangulation

Context

Discrete optimization problem :

$$\max\{c^\top x \colon Ax \leq b\}, \qquad (1)$$

where $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$ and $c \in \mathbb{Z}^n$.
Dual problem :

$$\min\{b^\top y \colon A^\top y = c, y \geq \mathbf{0}\}. \qquad (2)$$

Strong duality + integral properties
$\Downarrow$
combinatorial min-max theorem

(e.g. MaxFlow-MinCut, Ford & Fulkerson 1956)

TDI and box-TDI systems

A linear system $Ax \leq b$ is :

- **totally dual integral (TDI)** if $\forall c$, (2) has an integer optimum whenever it is finite.
- **box-TDI** when $Ax \leq b, u \leq x \leq v$ is TDI $\forall u, v \in \mathbb{Z}^n$.

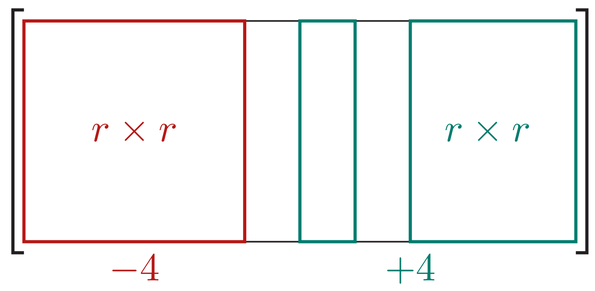
Totally unimodular (TU) matrix

A matrix is **TU** when all its nonzero subdeterminants equal ± 1.

Theorem (Hoffman, Kruskal, 1956)
A TU $\Leftrightarrow \forall b \in \mathbb{Z}^m, Ax \leq b$ is box-TDI.

A generalization of TU matrices

A full row rank matrix R of size $r \times n$ is **equimodular** when all its $r \times r$ nonzero subdeterminants have the same absolute value, denoted by $\mathrm{eqdet}(R)$.

$R =$ 

A matrix is **totally equimodular** when all its full row rank submatrices are equimodular.

Totally equimodular (TE) matrices : decomposition and triangulation

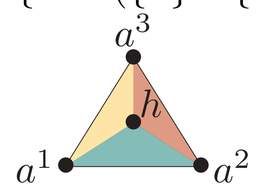
A set of vectors $A = \{a_1, \ldots, a_r\} \subseteq \{0, \pm 1\}^n$ is called a **te-set** if it is linearly independent and its associated matrix, formed by taking the vectors as rows, is totally equimodular (TE).

The **cone** generated by the set A is

$$C = \mathrm{cone}(A) = \{\textstyle\sum_{i=1}^r \alpha_i a_i \colon \alpha_i \geq 0\} \subseteq \mathbb{R}^n.$$

The **Hilbert basis** of C is the minimal set of integer vectors of C whose nonnegative integer combinations describe every integer vector of C.

A **unimodular Hilbert triangulation (UHT)** of C is a collection of cones generated by linearly indepedent Hilbert basis elements of C, which cover C, and none of them "overlap".

Type of te-set	$\mathrm{eqdet}(A)$	examples	Hilbert basis of $\mathrm{cone}(A)$	UHT
tu-set totally unimodular	1	$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$	A	trivial : $\{\mathrm{cone}(A)\}$
te-lace not a tu-set, every proper subset is a tu-set	2	$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$	$A \cup \{h = \frac{1}{2}\sum_i a_i\}$	stellar subdivision : $\{\mathrm{cone}(\{h\} \cup \{a_j\}_{j\neq i})\}_i$ 
te-interlace size ≥ 2 ($\Leftrightarrow$ not a tu-set) and every pair forms a te-lace	**thin :** 2^{r-1}	$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{bmatrix}$	$A \cup \{\frac{1}{2}(a_i + a_j)\}_{i\neq j}$	triangulation of the second hypersimplex
	thick : 2^r	$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$	$A \cup \{\frac{1}{2}(a_i + a_j)\}_{i\neq j} \cup H$, where $H = \begin{cases} \{\frac{1}{4}\sum_i a_i\}, \text{ or} \\ \{\frac{3}{4}a_i + \frac{1}{4}\sum_{j\neq i} a_j\}_i. \end{cases}$	only for $n = 4$ and 6

Theorem (Chervet, Grappe, V., 2025$^+$)

Let A be a set of r linearly independent $0,\pm 1$ vectors. Then :

1. A is a te-set if and only if it can be partitionned as follows.

$$A = \underbrace{U}_{\text{tu-set}} \sqcup \underbrace{L_1 \sqcup \cdots \sqcup L_k}_{\text{te-laces}} \sqcup \underbrace{S_1 \sqcup \cdots \sqcup S_\ell}_{\text{thin te-interlaces}} \sqcup \underbrace{T_1 \sqcup \cdots \sqcup T_m}_{\text{thick te-interlaces}}$$

   where $U, L_1, \ldots, L_k, S_1, \ldots, S_\ell, \ldots, T_1, \ldots, T_m$ are *mutually-tu*.
2. If moreover no T_i is of size 8 or more, then $\mathrm{cone}(A)$ has a regular unimodular Hilbert triangulation.

Corollary

Let A be a TE matrix, then $\forall b \in \mathbb{Z}^n$, the system $Ax \leq b$ is totally dual dyadic.

Conjecture

There is no thick te-interlace of size 8 or more.

Chervet, P., Grappe, R., Vallée, M., (2025). *Totally equimodular matrices : decomposition and triangulation (arXiv :2504.05930).*

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