Toric manifolds of Picard number 4

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Université Libre de Bruxelles[1]

Joint work with: Suyoung Choi and Hyeontae Jang

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Study of fans $\simeq$ Toric geometry

Toric variety: normal algebraic variety over the field of complex numbers $\mathbb{C}$ that admits an effective algebraic action of $(\mathbb{C}^*)^n$ having a dense orbit.

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Fundamental theorem of toric geometry

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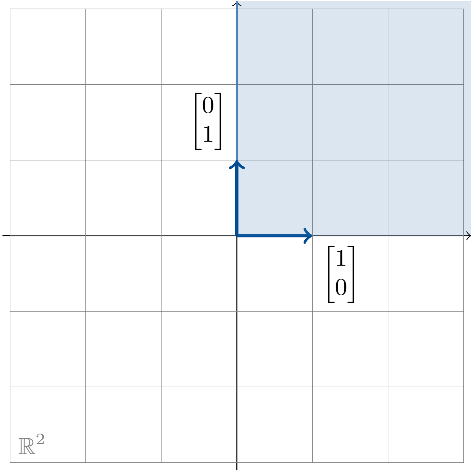
Smooth complete fan : Σ finite collection of cones C

A smooth complete fan in $\mathbb{R}^2$



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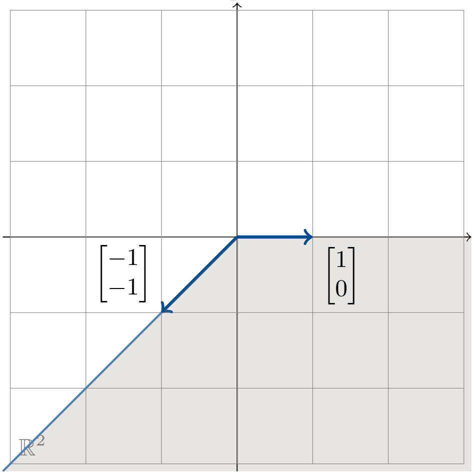

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Smooth complete fan (fan) : Σ finite collection of cones C stable by faces and intersection

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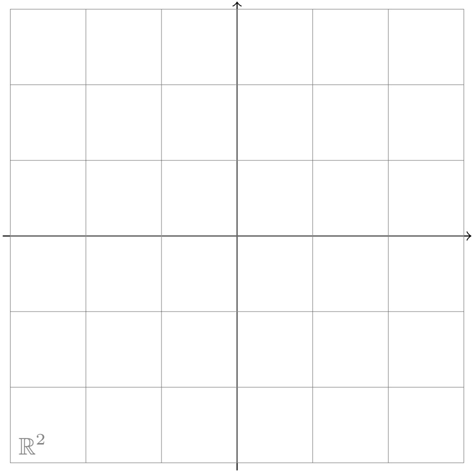

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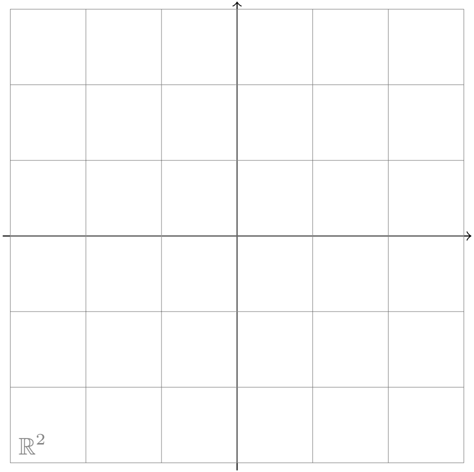

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(complete)	union is $\mathbb{R}^n$

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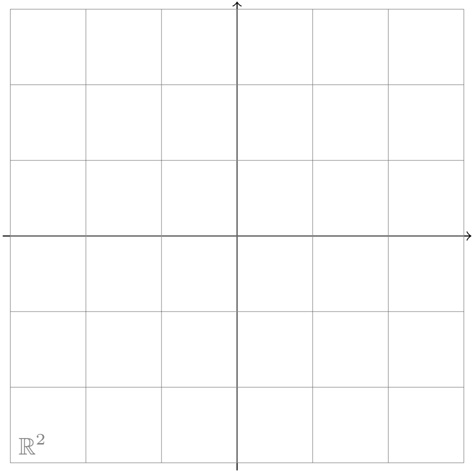

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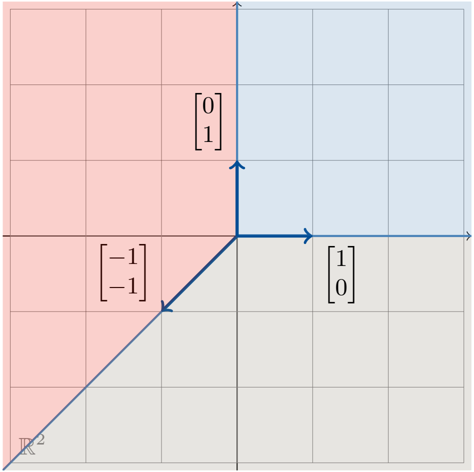

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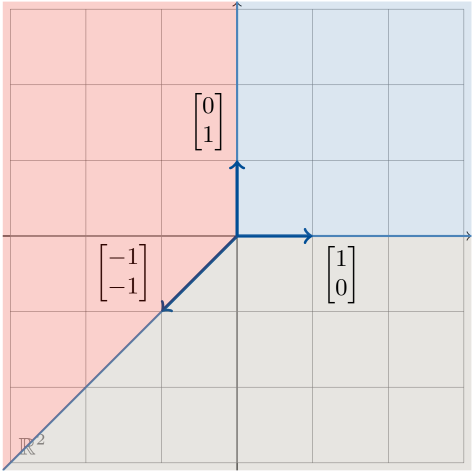

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Dimension of Σ: n

Picard number of Σ: $m - n$, m number of rays

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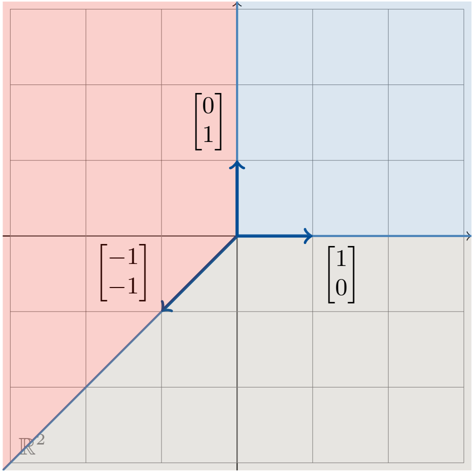

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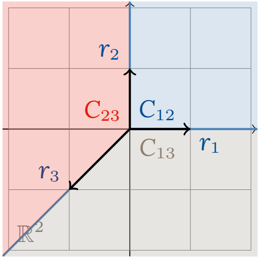

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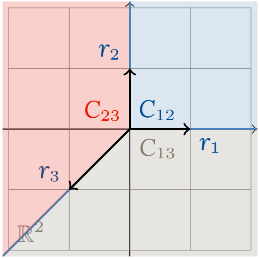

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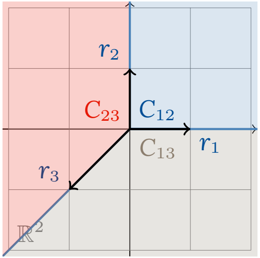
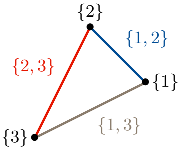


$$\longrightarrow$$

$K =$



$$\lambda = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \end{array}$$

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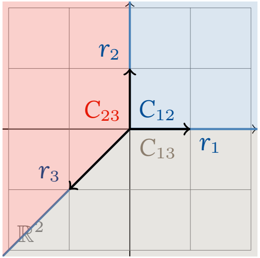
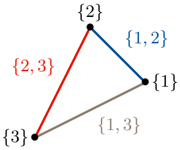


$\longrightarrow$ $K =$



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Facts

- K is a simplicial complex.

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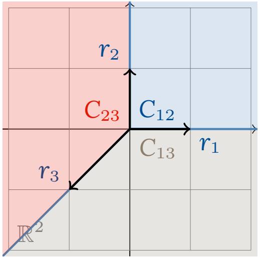
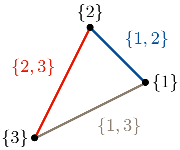


$\longrightarrow$ K =



$$\lambda = \begin{array}{c} \begin{matrix} \quad 1 & \quad 2 & \quad 3 \end{matrix} \\ \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \end{array}$$

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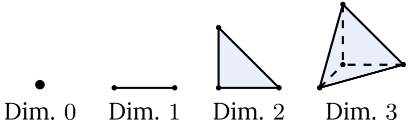
Simplices:



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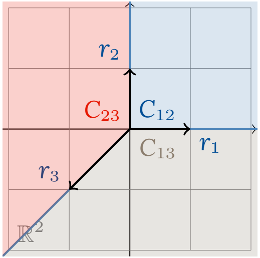
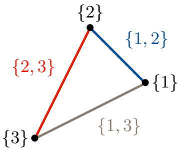


$\longrightarrow$ $K =$



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Facts

- K is a simplicial complex.
- K is pure.

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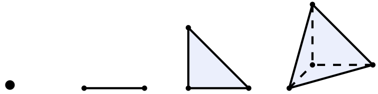

Dim. 0 Dim. 1 Dim. 2 Dim. 3

Our goal

Characterization problem in toric geometry

Enumerate every such pairs (K, λ), which correspond to fans Σ.

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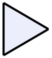
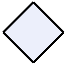
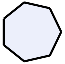
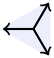
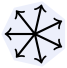
Polygon			
Normal fan			

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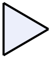
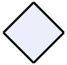
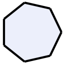
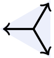
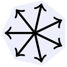
Polygon			
Normal fan			

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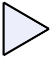
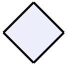
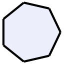
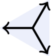
Polygon			
Normal fan			

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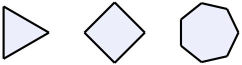
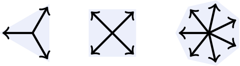
Polygon	
Normal fan	

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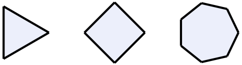
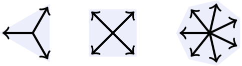
Polygon	
Normal fan	

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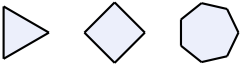
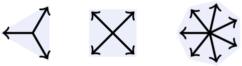
Polygon	
Normal fan	

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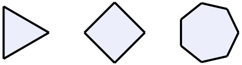
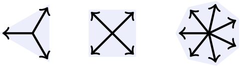
Polygon	
Normal fan	

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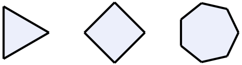
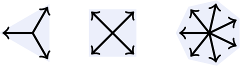
Polygon	
Normal fan	

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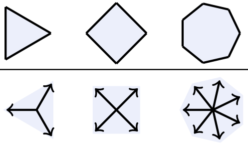
Polygon

Normal fan



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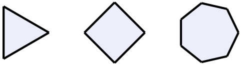
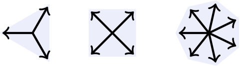
Polygon	
Normal fan	

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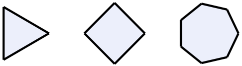
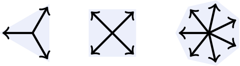
Our goal

Characterization problem in toric geometry

Enumerate every such pairs (K, λ), which correspond to fans Σ.

Two possible approaches:

1. Small dimension. (Ex. dimension 2)

Polygon	
Normal fan	

2. Small Picard number.
   - $\mathrm{Pic}\,\Sigma = 2$ (Kleinschmidt 88).
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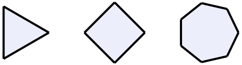
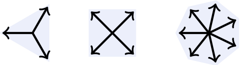
Polygon	
Normal fan	

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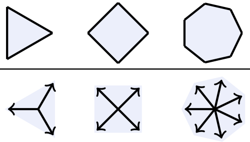

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→ Properties on K

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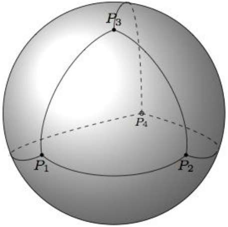

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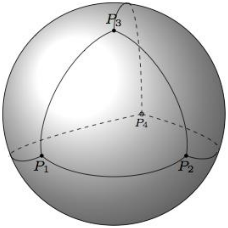

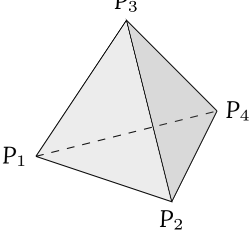

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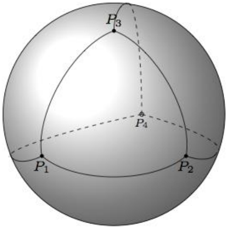

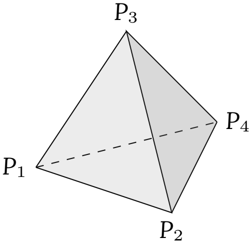

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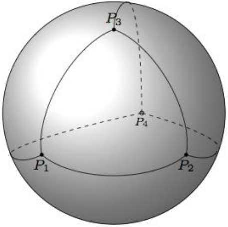

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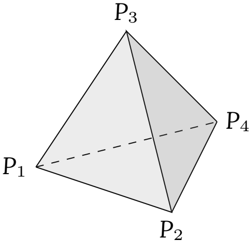

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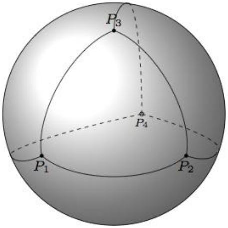

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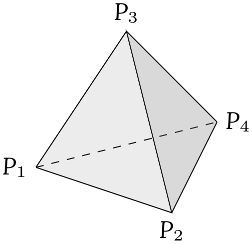

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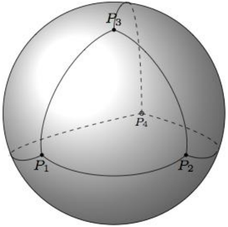

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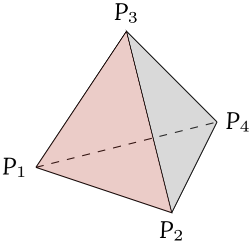

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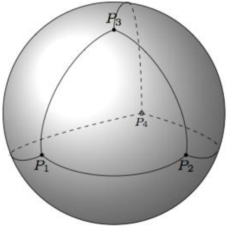

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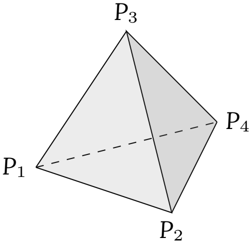

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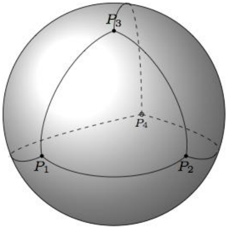

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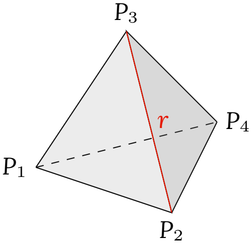

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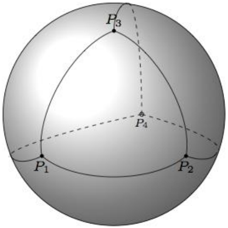

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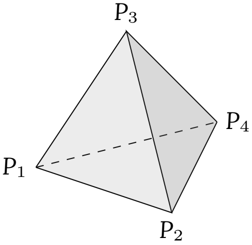

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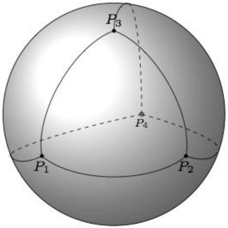

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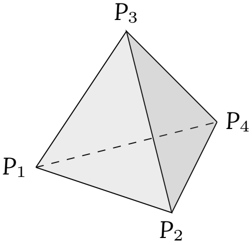

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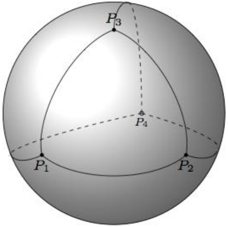

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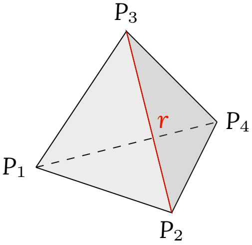

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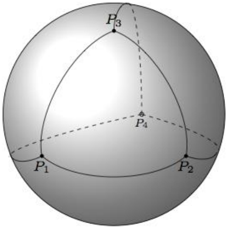

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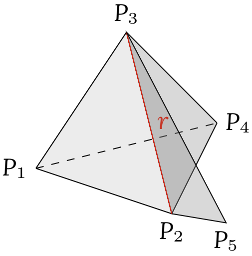

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Transpose of χ^{P_5}:

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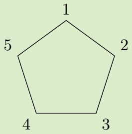

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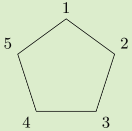

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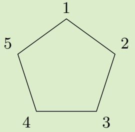

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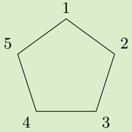

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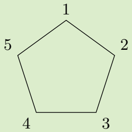

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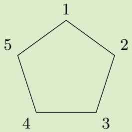

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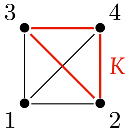

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$$
\begin{array}{cc} & \begin{array}{cccccc} 12 & 13 & 14 & 23 & 24 & 34 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 2 \\ 2 \end{bmatrix}
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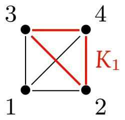

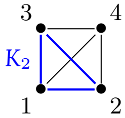

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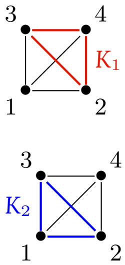

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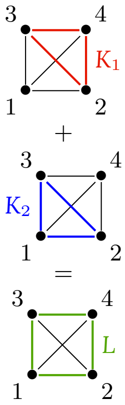

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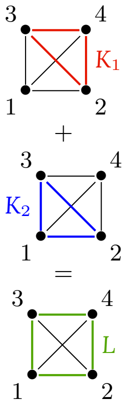

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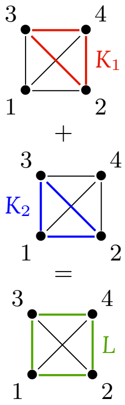

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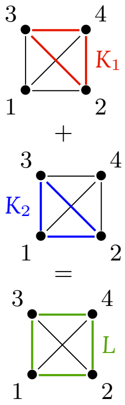

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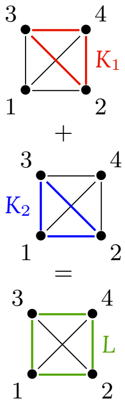

Remarks

1. Step 2 is highly parallel computing capable $\to$ use GPU!
2. This algorithm may return isomorphic weak pseudo-manifolds.
3. There is an infinite number of weak pseudo-manifolds of Picard number 4.

$\rightarrow$ Property on λ

$\Sigma \;\leftrightarrow\; (K, \lambda)$
Smooth complete fan

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$\xrightarrow{\text{smooth}}$

facet of K

$$\lambda = \begin{bmatrix} * & & * & \boxed{\begin{matrix} * & \cdots & * \\ \vdots & & \vdots \\ * & \cdots & * \end{matrix}} & * & & * \\ \vdots & \cdots & \vdots & & \vdots & \cdots & \vdots \\ * & & * & & * & & * \end{bmatrix}$$

$\Rightarrow \det = \pm 1$

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$$\lambda = \left[\begin{matrix} * \\ \vdots \\ * \end{matrix} \;\; \cdots \;\; \begin{matrix} * \\ \vdots \\ * \end{matrix} \;\; \overset{\text{facet of } K}{\boxed{\begin{matrix} * & \cdots & * \\ \vdots & & \vdots \\ * & \cdots & * \end{matrix}}} \;\; \begin{matrix} * \\ \vdots \\ * \end{matrix} \;\; \cdots \;\; \begin{matrix} * \\ \vdots \\ * \end{matrix}\right]$$

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$$\lambda = \begin{bmatrix} * & & * & \color{red}{*} & \color{red}{\cdots} & \color{red}{*} & * & & * \\ \vdots & \cdots & \vdots & \color{red}{\vdots} & & \color{red}{\vdots} & \vdots & \cdots & \vdots \\ * & & * & \color{red}{*} & \color{red}{\cdots} & \color{red}{*} & * & & * \end{bmatrix}$$

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mod 2

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$$\lambda^{\mathbb{R}} = \begin{bmatrix} 0/1 & & 0/1 & \color{red}{0/1} & \color{red}{\cdots} & \color{red}{0/1} & 0/1 & & 0/1 \\ \vdots & \cdots & \vdots & \color{red}{\vdots} & & \color{red}{\vdots} & \vdots & \cdots & \vdots \\ 0/1 & & 0/1 & \color{red}{0/1} & \color{red}{\cdots} & \color{red}{0/1} & 0/1 & & 0/1 \end{bmatrix}$$

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$\Rightarrow$ mod 2 base

$\rightarrow$

The **facets** of K are **bases** of some binary matroid.

$\rightarrow K$ is $\mathbb{Z}_2^n$-colorable

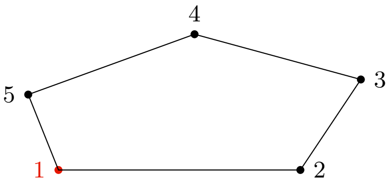
Wedge operation and seeds

Wedge operation and seeds

Definition (Wedge operation)

The wedge of K at v is the simplicial complex on $V \cup \{v_1, v_2\} \backslash \{v\}$:

$$\mathsf{Wed}_v(K) := (I * \mathsf{Lk}_K(v)) \cup (\partial I * K \backslash \{v\}).$$



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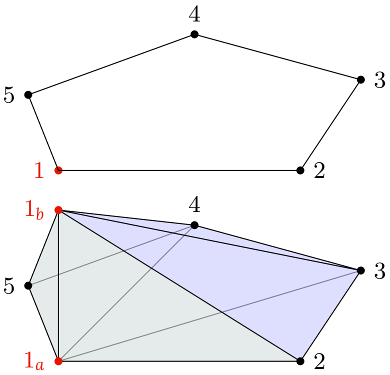

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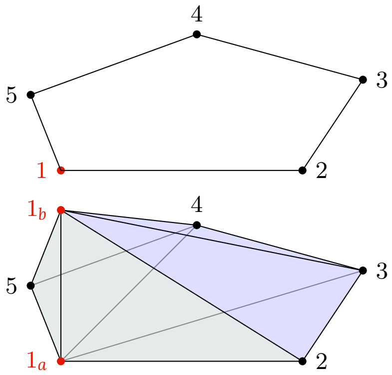
$\rightarrow$ simplicial complexes which are not wedges are called seeds.



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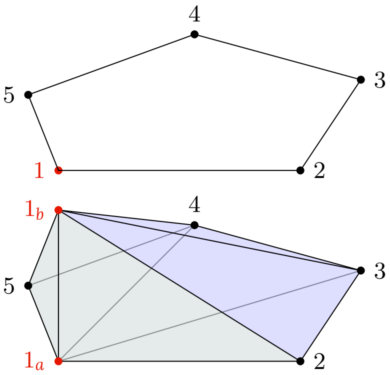
Consecutive wedges:

J-construction (Bahri, Bendersky, Cohen and Gitler, 2010)



Properties of wedged seeds

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In our case: Picard number 4

Finite number of cases:

$$(n, m) = (1,5), (2,6), (3,7), (4,8), (5,9), (6,10), (7,11), (8,12), (9,13), (10,14), (11,15),$$

$\rightarrow$ finite problem!

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First strategy

Run the GPU algorithm for $n \leq 11$ and check $\mathbb{Z}_2^n$-colorability.

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New strategy

Run the GPU algorithm with A the ridge-facet incidence matrix of every such binary matroids.

Keeping the right ones

Bases of such a binary matroid $\longrightarrow$ GPU algorithm

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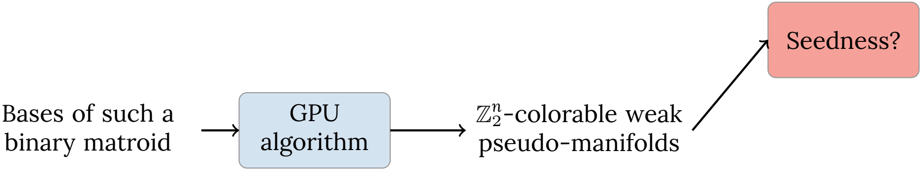
Bases of such a binary matroid $\longrightarrow$ $\boxed{\text{GPU algorithm}}$ $\longrightarrow$ $\mathbb{Z}_2^n$-colorable weak pseudo-manifolds

Keeping the right ones



Keeping the right ones

Bases of such a binary matroid → GPU algorithm → $\mathbb{Z}_2^n$-colorable weak pseudo-manifolds → Seedness?

Lemma (Seedness)

A simplicial complex K is a seed if and only if there is no face $\{v, w\}$ of K such that for every minimal non-face σ of K we have either $\sigma \cap \{v, w\} = \emptyset$ or $\{v, w\}$.

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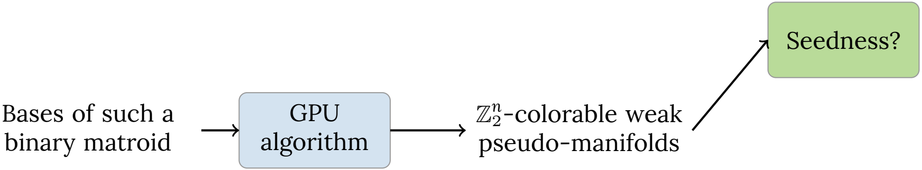

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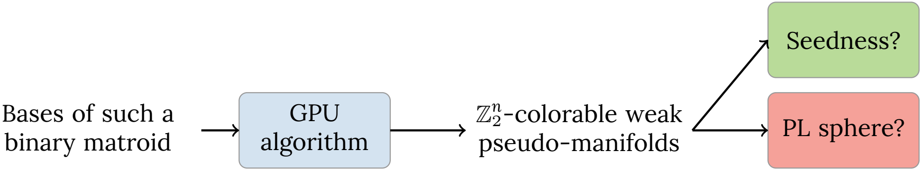

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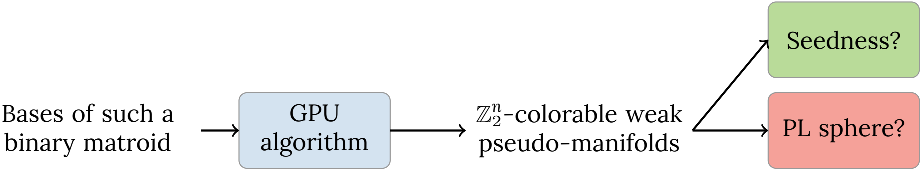

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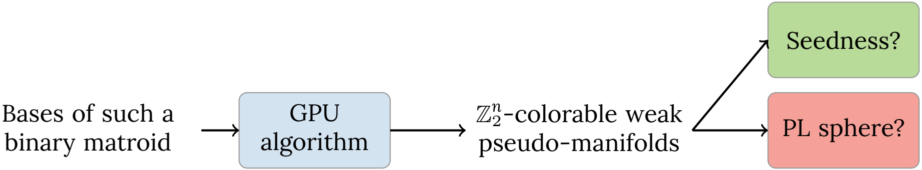

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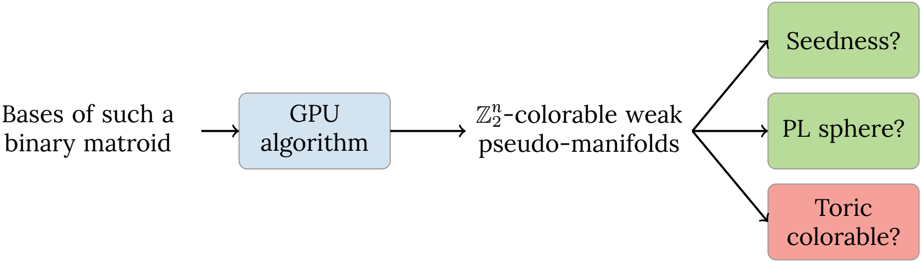

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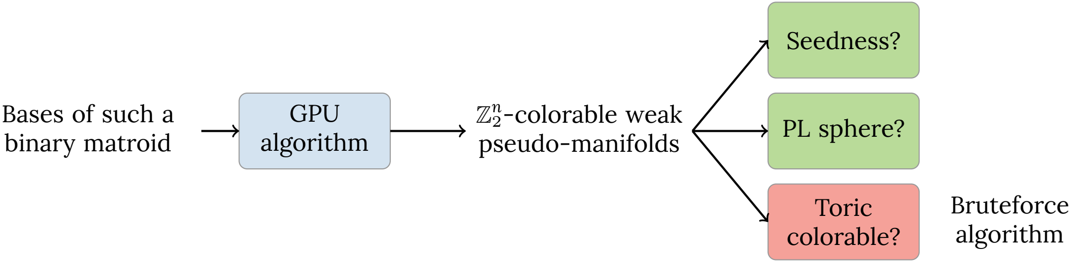

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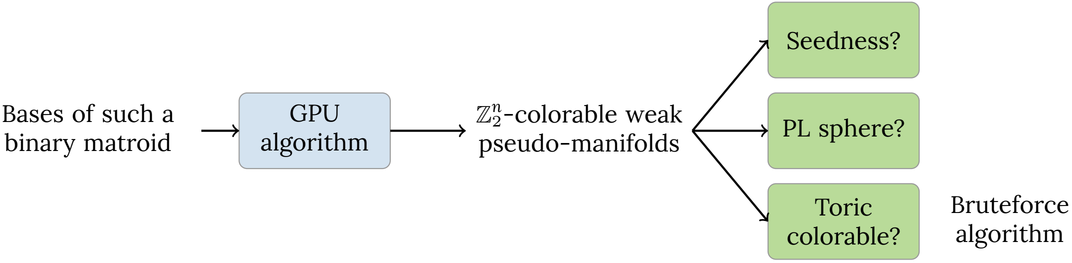

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Toric colorable seeds of Picard number 4

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The number of toric (or $\mathbb{Z}_2^n$-)colorable seeds of dimension $n-1$ and Picard number $p \leq 4$ is as follows:

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	n												
p	1	2	3	4	5	6	7	8	9	10	11	> 11	*total*
1	1												1
2		1											1
3		1	1	1									3
4		1	4	21	142	733	1190	776	243	39	4		3153

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The database is available online:

`https://github.com/MVallee1998/GPU_handle`

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Theorem (Choi, Jang, and V., 2025+)

The number of $(n-1)$-dimensional fanlike seeds with Picard number $p \leq 4$ is as follows:

	n								
p	2	3	4	5	6	7	8	> 8	total
2	1								1
3	1	1							2
4	1	4	10	16	18	9	1		59

Fanlike seeds of Picard number 4

This database is helpful for solving problems concerning toric manifolds of Picard number 4

- We answer a question asked by Chen, Fu and, Hwang in 2014 (Choi, Jang, V., 2024)
- We solve the lifting problem for that case (Choi, Jang, V., 2025)

Theorem (Choi, Jang, and V., 2025+)

The number of $(n-1)$-dimensional fanlike seeds with Picard number $p \leq 4$ is as follows:

	n								
p	2	3	4	5	6	7	8	> 8	*total*
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The database is available online:

`github.com/Hyeontae1112/toric_manifolds_with_Picard_number_4`

What remains to do

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What remains to do

- Find which ones are projective

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- Picard number 5...

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[1]

Thank you for listening!

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