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Cones, fans, and wedges : combinatorial and geometrical perspectives

Cones, éventails, et biseaux : perspectives combinatoires et géométriques

Thèse de doctorat

présentée par

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Spécialités

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Avant-propos

Si l'on m'avait donné des plumes,
je me serais sans doute envolé.
Une pioche m'aurait fait creuser,
un marteau et une enclume
m'auraient invité
à façonner l'acier.

Il en faut bien peu
pour occuper l'humain :
quelques outils dans les mains,
et le voilà heureux,
à fabriquer, inventer, imaginer.

À moi, on a confié
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Voici ce que j'en ai fait.

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m'ont offert ces outils
tout au long de ma vie.

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Une thèse qui débute.

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Cones, fans, and wedges : combinatorial and geometrical perspectives

Mathieu Vallée

9 octobre 2025

Abstract

This thesis focuses on the study of geometric-combinatorial objects called cones and fans, which arise notably in optimization and toric geometry. These objects live in real Euclidean space and are closely related to polyhedra and polytopes.

The first part of the thesis is motivated by problems in combinatorial optimization. The solutions to such problems are often given by the integer points in a polyhedron. We investigate the integer points in cones associated with problems satisfying a special property: box-TDI. This property enables strong combinatorial min-max theorems. Specifically, we analyze totally equimodular matrices, a generalization of totally unimodular matrices, and provide a decomposition theorem for their rows when they are linearly independent. Using this result, we determine the Hilbert basis of any cone generated by these rows. Furthermore, we construct a regular unimodular Hilbert triangulation for almost all such cones.

In the second part, we explore fans from the perspective of toric geometry and topology. A fan is a collection of cones in real Euclidean space that intersect only along common faces. In particular, any polyhedron gives rise to an associated fan. The fundamental theorem of toric geometry establishes a correspondence between toric varieties and fans. A central problem in this field is the classification of complete non-singular toric varieties, which correspond to complete non-singular fans. The Picard number of a fan is defined as the number of its rays minus its dimension. Smooth complete fans with Picard numbers 2 and 3 have been fully characterized by Kleinschmidt (1988) and Batyrev (1991). A fan is encoded by a fan-giving PL sphere and a finite set of generators. According to Choi and Park, any fan-giving PL sphere of fixed Picard number can be constructed from a finite set of minimal building blocks —called seeds— using the wedge operation. This inductive process is known as toric wedge induction. Using a massively parallel algorithm executed on a graphics card, we enumerate a superset of fan-giving seeds for the case of Picard number 4. This database is then used to solve the toric lifting problem in this context. We also study an algorithm related to toric wedge induction in the setting of real toric varieties.

Résumé

Cette thèse porte sur l'étude d'objets géométrico-combinatoires appelés cônes et éventails, qui apparaissent notamment en optimisation et en géométrie torique. Ces objets vivent dans l'espace euclidien réel et sont étroitement liés aux polyèdres et aux polytopes.

La première partie de cette thèse est motivée par des problèmes d'optimisation combinatoire. Les solutions de tels problèmes sont souvent données par les points entiers d'un polyèdre. Nous étudions alors les points entiers dans des cônes associés à des problèmes satisfaisant une propriété particulière : ils sont box-TDI. Cette propriété permet d'établir de puissants théorèmes min-max de nature combinatoire. Plus précisément, nous analysons les matrices totalement equimodulaires, une généralisation des matrices totalement unimodulaires, et établissons un théorème de décomposition pour leurs lignes, dans le cas où celles-ci sont linéairement indépendantes. À l'aide de ce théorème, nous déterminons la base de Hilbert de tout cône engendré par ces lignes. Nous construisons ensuite, pour presque tous ces cônes, une triangulation de Hilbert régulière unimodulaire.

Dans la seconde partie, nous explorons les éventails du point de vue de la géométrie et de la topologie toriques. Un éventail est une collection de cônes dans l'espace euclidien réel qui ne s'intersectent que selon des faces communes. En particulier, il est possible d'associer un éventail à tout polyèdre. Le théorème fondamental de la géométrie torique établit une correspondance entre les variétés toriques et les éventails. Un problème central dans ce domaine est la caractérisation des variétés toriques lisses et complètes, dont les éventails sont complets et non singuliers. Le nombre de Picard d'un éventail est défini comme le nombre de ses générateurs diminué de sa dimension. Les éventails complets et non-singuliers de nombres de Picard 2 et 3 ont été entièrement caractérisés par Kleinschmidt (1988) et Batyrev (1991). Un éventail est encodé par une sphère PL fan-giving et un ensemble fini de générateurs. Selon Choi et Park, toute sphère PL fan-giving de nombre de Picard fixé peut être obtenue à partir d'un ensemble fini de briques fondamentales minimales pour l'opération de biseaux : les graines fan-giving. Ce processus s'appelle une induction par biseau torique. À l'aide d'un algorithme massivement parallèle exécuté sur carte graphique, nous énumérons un sur-ensemble des graines fan-giving dans le cas du nombre de Picard 4. Nous utilisons ensuite cette base de données pour résoudre le problème de relèvement torique dans ce contexte. Nous étudions également un algorithme lié à l'induction par biseau torique dans le cadre des variétés toriques réelles.

Contents

Introduction

An introductory hors d'œuvre

Optimization aims to find the "best" solution to a given problem. Many problems in economics, industry, and logistics fall into this category, typically involving the minimization of costs while maximizing profits under resource constraints.

Consider the following Bretagne-inspired example, commonly used in introductory courses on linear optimization: A bakery owner has recipes for two Breton pastries, the "kouign amann" and the "gâteau breton", as shown in Figure 1.1 and detailed in Table 1.1.

A kouign amann. ©Président Professionnel

A gâteau breton. ©court bouillon

Figure 1.1 – Two Breton pastries.

Product	8 kouign amann	6 slices of gâteau breton	Available quantity
Ingredients	200g of butter	250g of butter	7kg
	200g of sugar	250g of sugar	7kg
	800g of flour	360g of flour	23kg
	10g of baking powder	10g of baking powder	320g
	550ml of water	-	$+\infty$
	-	6 eggs	100
Price for one	2.50€	2.30€	

Table 1.1 – Recipe, price, and ingredient stock of two deserts from Bretagne.

The bakery owner wants to determine the best quantity of each pastry to produce in order to maximize profit. This problem is a linear optimization problem that we can model as follows.

Let k be the number of kouign amann produced, and g be the number of gâteau breton slices produced. We want to maximize the following quantity:

$$2.5k + 2.3g,$$

subject to the following constraints on k and g:

$$\left\{\begin{array}{rcrll} \frac{0.2}{8}k & + & \frac{0.25}{6}g & \leqslant 7, & \text{(butter)} \\ \frac{0.2}{8}k & + & \frac{0.25}{6}g & \leqslant 7, & \text{(sugar)} \\ \frac{0.8}{8}k & + & \frac{0.36}{6}g & \leqslant 23, & \text{(flour)} \\ \frac{10}{8}k & + & \frac{10}{6}g & \leqslant 320, & \text{(baking powder)} \\ \frac{0.55}{8}k & & & \leqslant +\infty, & \text{(water)} \\ & & \frac{6}{6}g & \leqslant 100, & \text{(eggs)} \\ k & & & \geqslant 0, & \text{(non-negative number)} \\ & & g & \geqslant 0. & \text{(non-negative number)} \end{array}\right. \tag{1.1}$$

These constraints ensure that the bakery does not exceed its available resources while ensuring non-negative production quantities.

First, we draw on the plane $\mathbb{R}^2$ the set of all points (k, g) that satisfy the system (1.1), called the feasible region, see Figure 1.2. The feasible region, which is a hexagon in this case, is a

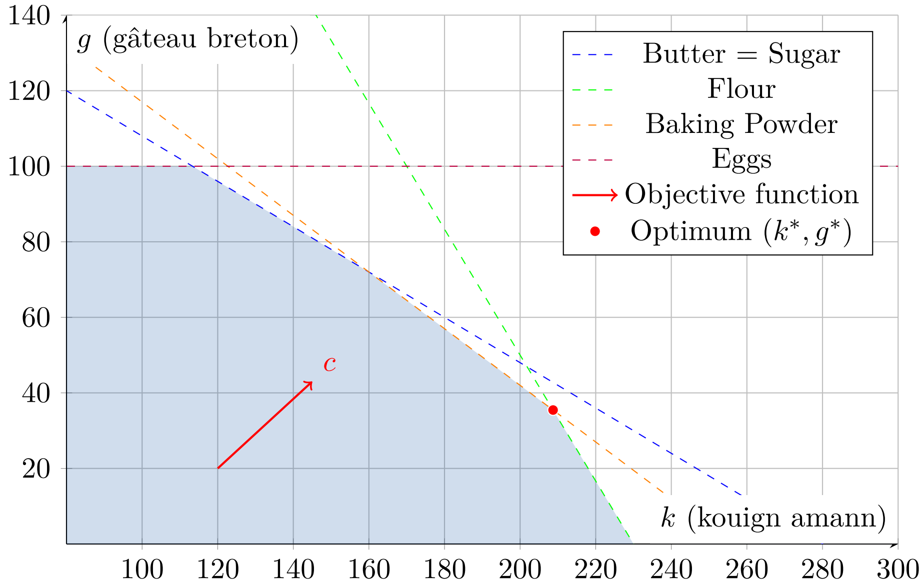


Figure 1.2 – The feasible region of the linear problem (1.1) is filled in light indigo.

two-dimensional polyhedron. Since it is bounded, it is a polytope.

Now, let us find a point (k^*, g^*) of this feasible region that maximizes the quantity $f(k, g) = 2.5k + 2.3g$, called the optimum. To do this, we consider the directing vector $c = (2.5, 2.3)$ of f, in red in Figure 1.2. This vector is the *gradient* of the linear function f, meaning that if we smoothly move in the feasible region using this direction, we increase the value of $f(k, g)$. Hence, when the value cannot be further increased, we end up on the boundary of the hexagon, and we can suppose we are at a vertex. However, note that this optimal vertex is approximately

(208.72, 35.46) and is not an integer point! It would be impractical to ask the bakers to prepare 208.72 kouign amanns and 35.46 slices of gâteau breton. In this thesis, this situation will never occur as we only consider integer polytopes, those whose vertices all have integer entries.

More generally, imagine the bakers have n recipes and m ingredients with finite quantity. In general, the feasible region becomes a polyhedron of dimension n defined with $m+n$ inequalities, as in (1.1). Drawing the feasible region is now quite difficult since it is of dimension n. Fortunately, computers are here to help us understand and compute these cases.

Historically, the first "computers" used to solve such optimization problems were humans using a computing typing machine. They used an approach developed by Dantzig called the simplex method. Intuitively, in the case of (1.1), we start at the base point $(0, 0)$ and jump to a neighboring vertex of the hexagon so it increases the value of the objective function. We repeat this process, jumping from vertex to vertex through the edges of the polytope. Note that at each step, we may have to choose one among the vertices increasing the objective function. This approach is very efficient in practice and is the one still being actively used nowadays, but it is now completely processed by machines. The number of steps required for the simplex method to converge is crucial for performance and depends heavily on the geometric properties of the polytope. This is one of the reasons why the combinatorics of polytopes attracts the attention of researchers.

Let us now introduce the two central objects through which this combinatorics will be explored: cones and fans. Given a finite set of vectors, the cone they generate is the set of their non-negative combinations, which can intuitively be seen as the region where there is light coming out of a flashlight. A fan is a collection of cones that "do not overlap". An example of fans arising from polytopes and which will be important in this thesis is the following. Let us consider a point on the boundary of the feasible region. The set of objective functions for which this point is optimal forms a cone. The normal fan of a polytope consists of these cones. In the case of the polytope described by the system (1.1), the normal fan is given in Figure 1.3.

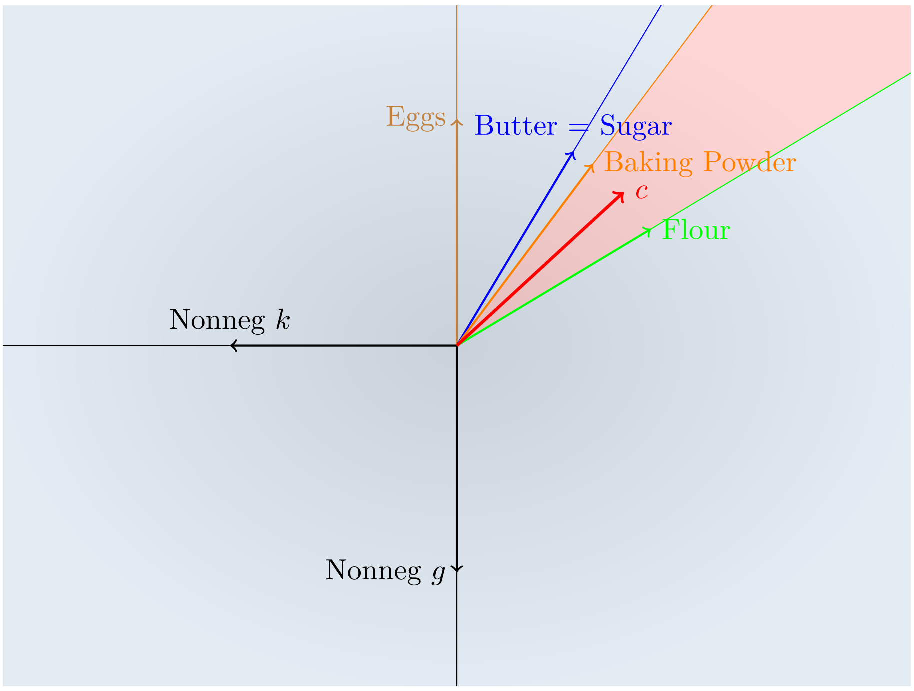


Figure 1.3 – The normal fan of the feasible set of the linear problem (1.1) and its (normalized) generators. The cone containing c is colored in red and the other five are colored in light indigo.

One should observe that the vector c is contained within the relative interior of a unique cone of this fan, which corresponds to the cone generated by the directing vectors of the inequalities from (1.1) that are satisfied with equality for the optimum point (k^*, g^*). Using this fan representation is quite useful because it allows us to directly determine which ingredients will be limiting when changing the prices of the pastries, i.e., when changing c.

The main challenge is that we often know very little about polytopes and their normal fans, especially as the number of variables (or recipes) increases, thereby raising the dimension. Additionally, in many cases there may be a large number of inequalities, making it impossible to fully describe the fan in practice.

The goal of this thesis is to systematically study polytopes via their associated cones and fans, with particular emphasis on the integer case.

Toric geometry is a branch of mathematics that studies objects called toric varieties, which correspond exactly to fans by the fundamental theorem of toric geometry. In this field, one focuses on the combinatorial structure of fans to derive geometric properties of toric varieties, and conversely. Throughout my thesis, I will focus on the case where polytopes of dimension n are simple. This means that each vertex of the polytope is connected to exactly n other vertices, for example, a cube is a simple polytope. In this context, the cones of the fan are simplicial, meaning they are generated by n linearly independent vectors. Even in this case, we have limited knowledge about the structure of these polytopes and fans. One of the most significant results in this area is the g-theorem, which provides an upper bound on the number of faces of each dimension for simple polytopes. This theorem was proved by Richard Stanley [127]. Given how difficult it is to understand the combinatorics of a general polytope, two approaches are used. In the first one, we consider polytopes with a small fixed dimension n. For example, it is common knowledge that polytopes and fans of dimension 2 correspond to polygons and fans of polygons, see Figure 1.4.

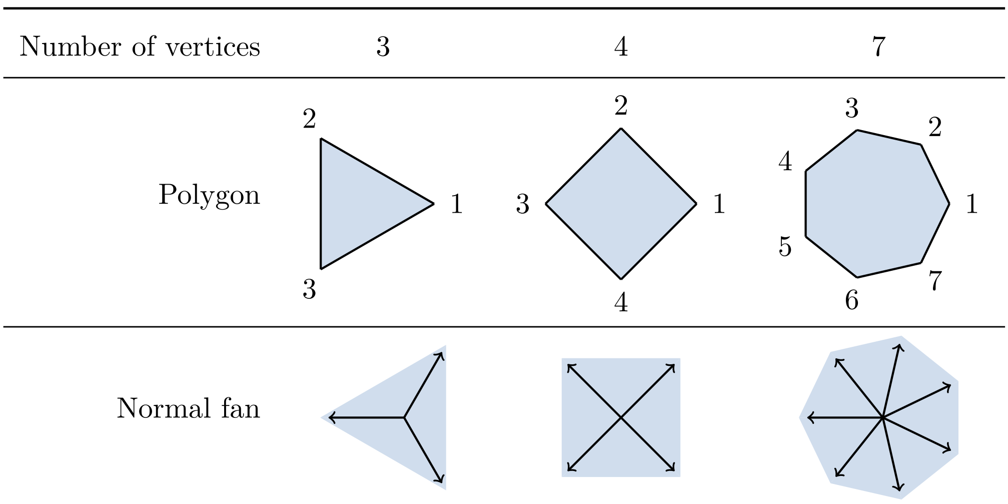


Figure 1.4 – Polygons are the 2-dimensional polytopes.

The second approach is based on considering polytopes and fans with a small fixed Picard number. The Picard number of a polytope is the difference between its number of facets and its dimension. There is only one type of polytope with Picard number 1 in each dimension n: the simplices, see Figure 1.5.

In this thesis, I contribute on this topic in two directions. The first part is a joint work with Patrick Chervet and Roland Grappe, in which we analyze simplicial cones called te-cones. These te-cones are generated by the row constraints of systems that belong to a bigger class known as box-TDI, that arise in combinatorial optimization. We obtain a decomposition theorem for this set of rows and utilize it to identify which rows should be added to generate every integer points of te-cones with non-negative integer combinations.

In the second part, I focus on fans with a small Picard number, which are furthermore non-singular: their cones are generated by parts of a basis of $\mathbb{Z}^n$. The case of Picard number 1 is straightforward: these are the normal fans of smooth simplices. A smooth simplex is, up to a unimodular transformation of the space, the convex hull of the origin $\mathbf{0}$ and the points $(1, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$, as shown in Figure 1.5. The classifications for com-

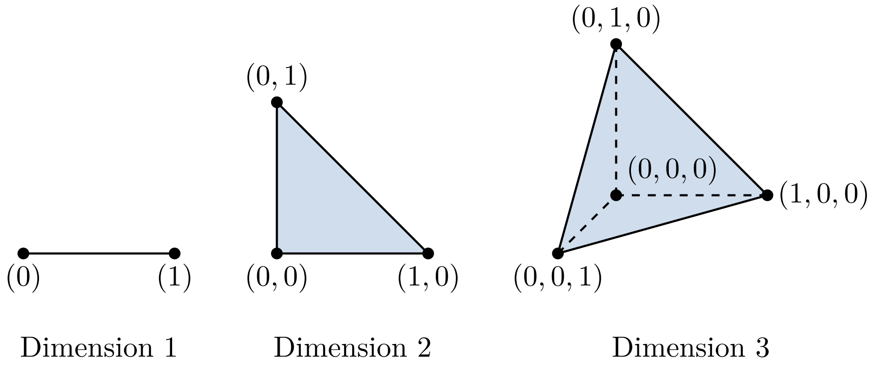


Figure 1.5 – Smooth simplices of dimension $n = 1, 2, 3$.

plete non-singular fans with Picard numbers 2 and 3 was completed by Kleinschmidt [99] and Batyrev [17], respectively. In the second part, we go one step further and focus on the case of Picard number 4. More precisely we include a joint work with Suyoung Choi and Hyeontae Jang in which we identify candidate combinatorial structures for such fans. The ultimate goal of this work is to characterize all smooth polytopes with Picard number 4.

After this quick tour on my thesis, we are now ready to delve into a more technical introduction.

Polytopes, linear optimization, and fans

A *linear optimization problem* consists in finding a point x^* which reaches:

$$\max\{c^\top x \colon x \in \mathbb{R}^n, Ax \leqslant b\}, \tag{1.2}$$

for A an $m \times n$ rational matrix and b a rational vector, and c a rational vector representing the *(linear) objective function* $x \mapsto c^\top x$. Note that A, b, and c can be chosen with entries in $\mathbb{Z}$, without changing the problem. The *feasible set* of this problem is the set $\{x \in \mathbb{R}^n \colon Ax \leqslant b\}$ and is a polyhedron. When the feasible set is bounded, it is a polytope P, and we know that x^* is in the boundary of P. In particular, it is reached by (at least) one vertex of P. The simplex method, first described by Dantzig, is a very efficient method in practice for finding such an optimum, and can be described as follows. One starts at any vertex v of P and jumps from one vertex to another along the edges of P, each time choosing an edge, called *a pivot*, which increases the objective function. Since P is bounded, this algorithm stops at a vertex v^* and provides an optimum. In other words, we construct a path from v to v^* in the 1-skeleton of P, where the 1-*skeleton* of P is the graph on the vertices and edges of P.

Another way of obtaining the optimum is by looking at the *dual problem* to (1.2):

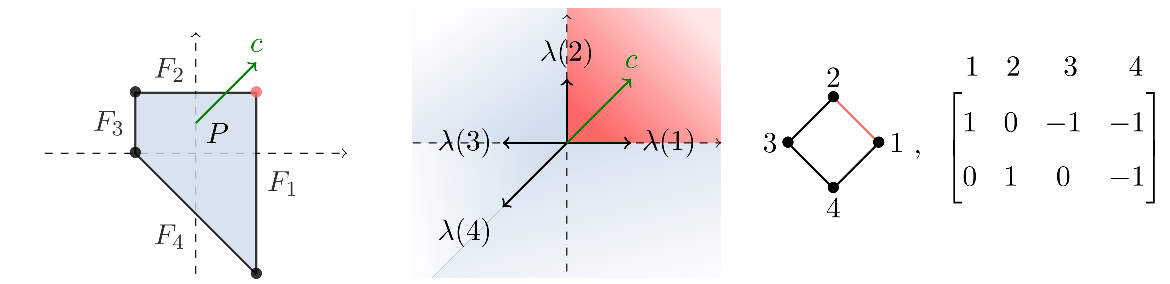
$$\min\{b^\top y \colon y \in \mathbb{R}^m, A^\top y = c, y \geqslant \mathbf{0}\}. \tag{1.3}$$

In fact, the *strong duality theorem* states that if (1.2) and (1.3) are finite, then they are equal and in that case we obtain the *strong linear programming duality*:

$$\max\{c^\top x \colon Ax \leqslant b\} = \min\{b^\top y \colon A^\top y = c, y \geqslant \mathbf{0}\}.$$

A row of A in the system $Ax \leqslant b$ is *active* for a face F of P if its associated inequality is an equality for any point in the relative interior of F. The *cone* generated by a finite set of rational vectors $R = \{r_1, \dots, r_k\}$ is the non-negative linear span of vectors of R, and is denoted by $\operatorname{cone}(R)$. When R is linearly independent, the cone is *simplicial*. The *normal fan* of P is the collections of cones spanned by the active rows of $Ax \leqslant b$ for every face of P. In particular, every cone of the fan is one-to-one associated with a face of P, when P is full-dimensional. The one-dimensional cones of the fan are called the *rays* and are in one-to-one correspondence with the facets of P.

In toric geometry, we split a fan into a combinatorial part and a geometrical part as follows. Every cone of a fan Σ of $\mathbb{R}^n$ is generated by the generators of its rays. Say that the rays of Σ are $\rho_1, \dots, \rho_m$, labeled by $[m] = \{1, \dots, m\}$, and let $\lambda \colon [m] \to \mathbb{Z}^n$ be a map such that ρ_i is generated by $\lambda(i) \in \mathbb{Z}^n$, for $i \in [m]$. Each cone C of Σ is associated with a subset $I \subseteq [m]$ such that $C = \operatorname{cone}(\{\lambda(i)\}_{i\in I})$. The *underlying complex* K of Σ is composed of all these subsets of $[m]$. The pair (K, λ) hence encodes the fan Σ. We often write λ as an $n \times m$ matrix whose i-th column is $\lambda(i)$. See Figure 1.6 for an example.



A polytope $P = \{x \colon Ax \leqslant b\}$ Its normal fan Its associated pair (K, λ)

Figure 1.6 – The combinatorial objects arising from a polytope $P = \{x \colon Ax \leqslant b\}$. The red vertex of P is maximal for the objective function $x \mapsto c^\top x$, with $c^\top = [1\ 1]$, and corresponds to the red cone in its normal fan, that contains c, and to the red face of its underlying complex $K = \{\varnothing, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,4\}, \{2,3\}, \{3,4\}\}$. In that case we have $\lambda = A^\top$ and $b^\top = [1\ 1\ 1\ 1]$. The 1-skeleton of P is in black on the left.

Therefore, the strong duality theorem can be restated as follows for polytopes: *"finding an optimal vertex of P for $x \mapsto c^\top x$ is essentially equivalent to finding a cone of the normal fan of P that contains c".* Indeed, once this cone is found, we can directly compute the vertex associated to this cone as its generators are the inequalities that are active for an optimal vertex. Thus, the study of the normal fan of polytopes is of major importance in linear optimization.

Combinatorial optimization

In combinatorial optimization, we want to optimize an objective function over discrete objects. In most cases we can embed our discrete objects as integer points of $\mathbb{R}^N$, for some N.

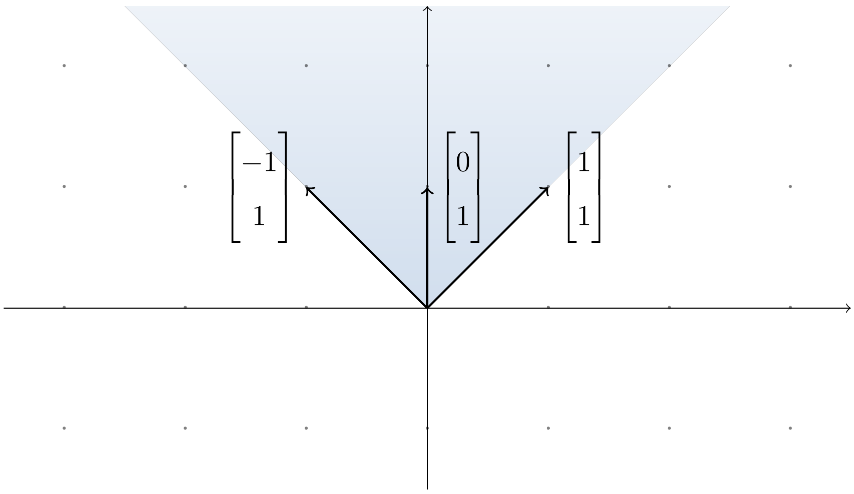


Figure 1.7 – The cone C generated by the two vectors $[1\ 1]^\top$ and $[-1\ 1]^\top$. Its Hilbert basis is composed of $[1\ 1]^\top, [-1\ 1]^\top$, and $[0\ 1]^\top$. Indeed, the vector $[0\ 1]^\top$ cannot be expressed as a non-negative integer combinations of any other integer vectors of C.

One way which covers numerous cases is by using what we call characteristic vectors. Given a finite set X, the canonical basis of $\mathbb{R}^X$ is $\{e_x\}_{x\in X}$. Let $X' \subseteq X$, its *characteristic vector* is the 0,1-vector $\sum\limits_{x\in X'} e_x \in \{0,1\}^X$.

In integer linear programming we consider the integer points in a polytope by adding the *integrality constraints* $x \in \mathbb{Z}^n$ to the problem (1.2). Adding these constraints not only destroys the strong duality but also makes the resolution of this problem way harder: solving a linear optimization problem is polynomial (by the ellipsoid method [81]) while solving an integer optimization problem is NP-complete [97]. Here, we suppose that the feasible set of (1.2) is integer, hence, there is no need for integrality constraints and the strong duality still holds. Nevertheless, there might not be integer optimal solutions in (1.3). In order to derive a combinatorial min-max theorem, like the MaxFlow-MinCut theorem [71], it would be desirable to have integer optimal solutions in (1.3), whichever integer objective function c is chosen. Systems which satisfy this property are called *totally dual integral (TDI)*, a notion closely related to Hilbert bases.

In fact, in the dual problem (1.3), finding a cone of the fan which contains c is not enough, since we now need to express c as a non-negative integer combination of the generators of this cone. A *Hilbert basis* of a cone C is an inclusionwise minimal set of integer vectors whose non-negative integer combinations generate every integer points of C, see Figure 1.7 for an example.

If the active rows at every vertex of the polytope is a Hilbert basis for the cone they generate, then, any integral c is in some cone generated by a Hilbert basis so it is a non-negative integral combination of the elements in this Hilbert basis. This yields an integer solution of (1.3) for every integer c, hence the system is TDI.

A system $Ax \leqslant b$ is *box-totally dual integral* [64] (*box-TDI*) if $Ax \leqslant b$, $\ell \leqslant x \leqslant u$ is TDI for all rational vectors ℓ and u (with possible infinite entries). TDI and box-TDI systems were introduced in the late 1970's and serve as a general framework for establishing various min-max relations in combinatorial optimization [122]. General properties of such systems can be found in [54], [122, Chap. 5.20], and [121, Chap. 22.4]. Two famous examples of box-TDI systems are the systems behind the MaxFlow-MinCut theorem of Ford and Fulkerson [71] and Kőnig's theorem about matchings in bipartite graphs [101].

More precisely, the box-TDIness of the two latter systems comes from the total unimodularity of the underlying matrix. A matrix is *totally unimodular* when all its subdeterminants are 0,±1. Totally unimodular matrices can be characterized in terms of box-TDI systems as follows.

Theorem 1.1 (Hoffman and Kruskal [91]). *A matrix A of $\mathbb{Z}^{m\times n}$ is totally unimodular if and only if the system $Ax \leqslant b$ is box-TDI for all $b \in \mathbb{Z}^m$.*

Then, the total unimodularity of incidence matrices of directed graphs and that of bipartite graphs yields the two aforementioned theorems.

Although every polyhedron can be described by a TDI system [121, Theorem 22.6], not every polyhedron can be described by a box-TDI system. *Box-TDI polyhedra* [54] are those that can be described by a box-TDI system.

In the context of box-TDI polyhedra, a generalization of totally unimodular matrices arises naturally: totally equimodular matrices [34]. An equivalent definition of total unimodularity is to ask for every set of linearly independent rows to be unimodular, where an $m \times n$ matrix is *unimodular* if it has full row rank and all its nonzero $m\times m$ determinants are ± 1. More generally, an $m \times n$ matrix is *equimodular* if it has full row rank and all its nonzero $m \times m$ determinants have the same absolute value. Then, *totally equimodular* matrices are those for which every set of linearly independent rows forms an equimodular matrix.

It turns out that totally equimodular matrices fulfill the same role for box-TDI polyhedra as totally unimodular matrices do for box-TDI systems.

Theorem 1.2 (Chervet, Grappe, and Robert [34, Corollary 8]). *A matrix A of $\mathbb{Q}^{m\times n}$ is totally equimodular if and only if the polyhedron $\{x : Ax \leqslant b\}$ is box-TDI for all $b \in \mathbb{Q}^m$.*

The question of their recognition is raised in [34, Open problem 1], and [79, Open problem 3.24] asks whether there exists a decomposition theorem for them.

Contributions. In the first part of Chapter 3, which comes from a submitted work with Patrick Chervet and Roland Grappe [35], we further investigate the connections between totally unimodular and totally equimodular matrices, and provide two new parallels: one in terms of pivots and trims, see Theorem 3.5.2, and another involving the transpose and the inverse. Then, we give our first main result: a decomposition theorem of totally equimodular matrices of full row rank, see Theorem 3.5.6. As a consequence, this provides a unified framework in which well-known classes of matrices appear intertwined, such as minimally non-totally unimodular, complement totally unimodular, and complement minimally non-totally unimodular matrices. Incidentally, linear systems associated with totally equimodular matrices are totally dual dyadic, see [1] for the definition.

Integer decomposition properties

An integer polytope $P \subseteq \mathbb{R}^n$ has the *integer decomposition property (IDP)*, or is *normal*, if for every positive integer dilation factor k, every integer point of $kP = \{kx \colon x \in P\}$ can be expressed as the sum of k integer points of P. Note that the IDP is also defined for general polyhedra, see [122]. Equivalently, P has the IDP if the Hilbert basis of the cone generated by $P \times \{1\} \subseteq \mathbb{R}^{n+1}$ is composed of the integer points of $P \times \{1\}$.

Let $P \subseteq \mathbb{R}^n$ be an integer polytope and L be the sublattice of $\mathbb{Z}^n$ spanned by the elements of the form $x - y$, where x and y are integer points of P. Then, P is *normal* when for every integer dilation factor k, every point of $kP \cap L$ can be expressed as the sum of k point of $P \cap L$, see [109, 22, 110, 86]. The IDP coincides with normality when L is generated by a part of a basis of $\mathbb{Z}^n$.

By Carathéodory's theorem [31], each point of a pointed (rational polyhedral) cone C is the non-negative combination of at most $\dim(C)$ generators of C. In combinatorial optimization, a preferable property is the so-called integer Carathéodory property. A cone C has the *integer Carathéodory property* (*ICP*) when each of its integer points can be expressed as a non-negative integer combination of at most $\dim(C)$ Hilbert basis elements. This property coincides with the integer Carathéodory property for polytopes [78] in the following sense: a polytope P has the integer Carathéodory property if and only if the Hilbert basis of the cone $C = \text{cone}(P \times \{1\})$ consists of the lattice points in $P \times \{1\}$ and C has the integer Carathéodory property.

As previously mentioned, Hilbert bases play a fundamental role in combinatorial optimization, as they underlie the notion of TDI systems [121, Theorem 22.5]. The integer Carathéodory property then ensures that an optimal solution of the dual problem is sparse [55]. Unfortunately, finding a Hilbert basis of a cone is hard in the generic case [114].

Over the years, a hierarchy of increasingly stronger integer decomposition properties has emerged: having a *unimodular Hilbert cover* (*UHC*), a *unimodular Hilbert triangulation* (*UHT*), and a *regular unimodular Hilbert triangulation* (*RUHT*). We have

$$\text{RUHT} \implies \text{UHT} \implies \text{UHC} \implies \text{ICP}.$$

Backgrounds about these properties can be found in [70, 24, 23, 21, 78], as well as the fact they strictly imply one another. Like the integer Carathéodory property, all these properties have polyhedral counterparts [87, Section 1.2.5].

Notably, Sebő [123] proved that cones of dimension at most three have a unimodular Hilbert triangulation. Matroid base polytopes satisfy the integer Carathéodory property [78], and it was recently shown that they admit a regular unimodular Hilbert triangulation [10], although this triangulation is nonexplicit. There exists a nonsimplicial cone for which no unimodular Hilbert triangulation is regular [70]. If a simplicial cone [102] or its dual [4] has small determinants, then it has the integer Carathéodory property. Incidentally, Kuhlmann asks [102]: *Does every simplicial cone have the integer Carathéodory property?* A related notion studied in [4] and [65] is the *integer Carathéodory rank*, which is the maximum over each of its integer vectors of the minimum number of its Hilbert basis elements needed to generate the vector.

As for polytopes, in 1997 at the Oberwolfach Conference "Combinatorial Convexity and Algebraic Geometry", documented in [109], Oda asks: *Does all integer smooth polytopes have the integer decomposition property?* From then, the research area has become quite active and we know that the following classes of polytopes satisfy the integer decomposition property: smooth polygons of $\mathbb{R}^2$, they trivially admit a unimodular triangulation, smooth centrally symmetric polytopes in $\mathbb{R}^3$ [18], smooth polytopes with totally unimodular constraint matrix [66], smooth reflexive polytopes of dimension $\leqslant 7$ [87, Theorem 3.37], polytopes with long edges [84], some Gelfand-Tsetlin polytopes [3], and smooth polytopes with Picard number at most 3 [116]. Note that for the latter case, the author used the classification of smooth projective fans of Kleinschmidt [99] and Batyrev [17]. Among them, some even satisfy stronger integer decomposition properties, find an overview in [87].

Contributions. We call *te-cones* the simplicial cones whose generators form the rows of a totally equimodular matrix of full row rank. As a consequence of the decomposition theorem mentioned in the previous section, we first derive the Hilbert basis of te-cones, see Theorem 3.6.3. Building on this, we construct an explicit regular unimodular Hilbert triangulation for most te-cones, with a well-described combinatorial structure, see Theorem 3.6.6. Supported by computational experiments, we conjecture that the untreated cases do not exist. This is done in the second part of Chapter 3, and part of our work with Patrick Chervet and Roland Grappe [35].

This study improves results on integer decomposition properties of simplicial cones as te-cones have arbitrary dimension as well as unbounded determinant.

Toric geometry

Fans are central object in the theory of toric geometry. A fan Σ is a collection of cones such that: (i) for every cone $C \in \Sigma$, the faces of C are in Σ, and (ii) for every two cones $C_1, C_2 \in \Sigma$, their intersection $C_1 \cap C_2$ is a face of each, and is therefore in the fan, by (i). Just as we did for normal fans of polytopes, every fan of $\mathbb{R}^n$ is associated to a pair (K, λ), where K is its underlying complex on the vertex set $[m]$ and $\lambda\colon [m] \to \mathbb{Z}^n$ is a map assigning a generator to each ray of Σ. The fundamental theorem of toric geometry states that the category of normal toric varieties is equivalent to that of rational pointed fans. Hence, combinatorial properties on a fan are translated into geometrical properties on its associated toric variety, and conversely. A fan is *complete* when it covers the ambient space. In that case the toric variety is also *complete*. A toric variety is *projective* when it comes from the normal fan of a polytope. A fan is *non-singular*, or *smooth*, if all its cones are non-singular, that is, spanned by a part of a basis of $\mathbb{Z}^n$. In particular, non-singular fans are simplicial. In that case the associated toric variety is smooth, and the complex K is a simplicial complex which is a *piecewise linear (PL) sphere*. A toric manifold is a smooth complete toric variety. In particular projective toric manifolds come from the normal fan of a smooth, or Delzant, polytopes. The *Picard number* of an n-dimensional fan of $\mathbb{R}^n$ with m rays is $m - n$.

One celebrated result using this correspondence between fans and toric manifolds is the proof of the g-conjecture by Stanley for polytopal spheres [127]. More precisely, by relating the number of faces of a polytope to the dimension of the cohomology ring of the associated toric variety, and then, leveraging geometrical properties satisfied by such manifolds, such as the Poincaré duality and the Hard Lefschetz theorem, this yields properties on the sequence of the number of faces of the polytope. More recently, Adiprasito proved the result for any PL sphere [2].

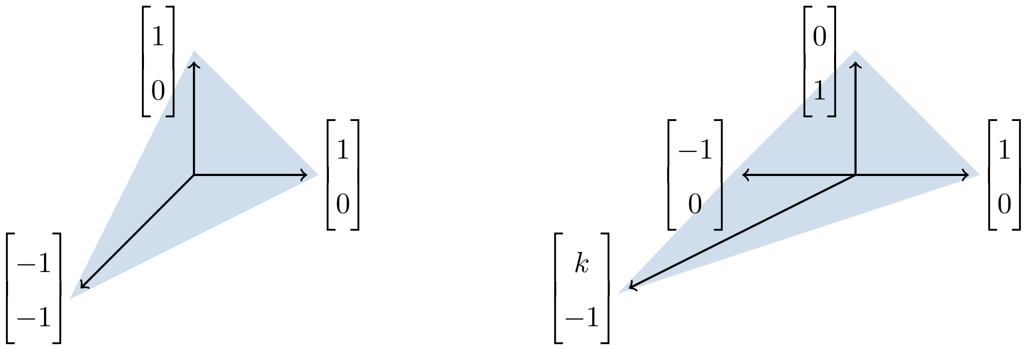
The best starting point to understand fans, and therefore toric manifolds, is to focus on those with small dimension, namely toric surfaces, those with small Picard number, or those corresponding to the normal fans of specific combinatorial polytopes. For that purpose, one needs to understand which PL spheres are good candidates to serve as their underlying complexes.

Two-dimensional complete non-singular fans are all projective and easily characterized since they all are normal fans to smooth polygons. In particular, they are obtained from successive blowups, namely subdividing cones of the fan, starting from $\mathbb{CP}^2$, the complex projective space of dimension 2, (the normal fan of a smooth triangle) or from a Hirzebruch surface [89] (the normal fan of a smooth right trapezoid), see [73]. These fans are illustrated in Figure 1.8.

The complete non-singular fans with Picard number 1 encode $\mathbb{CP}^n$, the complex projective space of dimension n, and are normal fans to a Delzant n-simplex. The complete non-singular fans with Picard number 2 and 3 are characterized by Kleinschmidt [99] and Batyrev [17], respectively, and are all normal fans to a polytope. In fact, PL spheres with Picard number at most 3 are all polytopal as shown by Mani [105]. Therefore, they can be described using Gale diagrams [82], whose dimension is one less than the Picard number. The characterization for Picard number at least 4 is not known. In fact, 3-dimensional Gale diagrams are complicated object to study. Moreover, there exists a non-polytopal PL sphere with Picard number 4 [83], and a complete non-singular fan whose underlying PL sphere can be non-polytopal [131].

Characteristic pairs Here, we write $\mathbb{Z}_2$ for the field on two elements $\{0, 1\}$ [1]. Deciding if a pair (K, λ) describes a fan requires some machinery for checking that the cones do not overlap,

1. We apologize to number theorists, to whom this notation refers to 2-adic integers.



The normal fan of a smooth triangle The normal fan of a smooth right trapezoid of type k

Figure 1.8 – The fans corresponding to the base toric manifolds of dimension 2.

see [96, 44] for instance. That is why toric topologists first work on a weakened version of pairs (K, λ) called characteristic pairs. A *characteristic pair* (K, λ) is composed of a PL sphere K on $[m]$ with $\dim(K) = n-1$ and a map $\lambda\colon [m] \to \mathbb{Z}^n$ which satisfies the *non-singularity condition* for K: $\forall \sigma \in K, \{\lambda(i)\}_{i\in\sigma}$ is part of a basis of $\mathbb{Z}^n$. In that case, we say that λ is a *characteristic map* over K. Moreover, when K supports a characteristic map, we say it is *toric colorable.* Note that the "bunch of cones" $\{\mathrm{cone}(\{\lambda(i)\}_{i\in\sigma})\colon \sigma \in K\}$ may not form a fan, however all its cones are non-singular: their Hilbert basis is the set of their generators. A *mod* 2 *characteristic pair* $(K, \lambda^{\mathbb{R}})$ [2] is composed of a PL sphere K on $[m]$ and of a map $\lambda^{\mathbb{R}}\colon [m] \to \mathbb{Z}_2^n$ which satisfies the *mod* 2 *non-singularity condition* for K: $\forall \sigma \in K, \{\lambda^{\mathbb{R}}(i)\}_{i\in\sigma}$ is mod 2 independent. In that case, we say that $\lambda^{\mathbb{R}}$ is a *mod* 2 *characteristic map* over K. Moreover, if K supports a mod 2 characteristic map, we says it is $\mathbb{Z}_2^n$*-colorable.* Note that if λ is a characteristic map over K, then $\lambda^{\mathbb{R}} := \lambda \pmod 2$ is a mod 2 characteristic map over K. Therefore, when K is toric colorable, it is $\mathbb{Z}_2^n$-colorable. When a mod 2 characteristic map over K comes from the mod 2 reduction of a normal fan of a polytope, the characteristic pair $(K, \lambda^{\mathbb{R}})$ is called a *small cover.*

Contributions. In Chapter 4, we find every toric colorable PL sphere with Picard number 4, solving the first step of the characterization of toric manifolds for that case. More precisely, these toric colorable PL spheres can be produced from a finite set of PL spheres called seeds by the means of successively using a specific operation on simplicial complexes, that is stable in the category of PL spheres, called the (simplicial) wedge operation. The number of such toric colorable seeds with Picard number at most 4 is given in Theorem 4.1.2. The main approach used here is a GPU algorithm to enumerate every $\mathbb{Z}_2^n$-colorable seeds, as it was proved by Choi and Park that their number is finite when fixing the Picard number. These results come from an article published in 2024 in Crelle's journal [39].

Toric wedge induction

A novel and powerful approach for obtaining results on toric manifolds with small Picard numbers is the one of Choi and Park, developed in [44], which uses the so-called wedge operation. This method allowed them to reprove the characterization of toric manifolds with Picard number at most 3 in [46].

The *wedge operation* first appeared in combinatorial optimization for tackling problems about

2. The $\mathbb{R}$ exponent notation comes from the fact that mod 2 characteristic maps appear in the context of *real* toric manifolds.

diameters of polytopes, see [98]. It has been translated for simplicial complexes in [115]. Notably, a construction using the simplicial wedge operation was used to disprove the Hirsch conjecture, see [119]. Let K be a simplicial complex on the vertex set $[m]$ and $v \in [m]$ be a vertex of K, the simplicial wedge $\mathrm{wed}_v(K)$ of K at v is a PL sphere with an additional copy of v and of dimension $\dim(K)+1$ intuitively obtained by inserting a segment into v in K. In particular, it has the same Picard number as K, and it contains two copies of K as subcomplexes, see Figure 1.9. For the complete definition, see Chapter 2. The wedge operation can be repeated

The boundary of a pentagon $\mathcal{P}_5$ The wedge of a pentagon at a vertex

Figure 1.9 – Illustration of a simplicial wedge of the boundary of a pentagon. The facets of the pentagon are $\{1,2\},\{1,5\},\{2,3\},\{3,4\}$, and $\{4,5\}$. The facets of the wedge at the vertex 1 are $\{1_1,1_2,2\},\{1_1,1_2,5\},\{1_1,2,3\},\{1_1,3,4\},\{1_1,4,5\},\{1_2,2,3\},\{1_2,3,4\}$, and $\{1_2,4,5\}$. The two copies of K in $\mathrm{wed}_1(K)$ have vertices $1_i,2,3,4$, and 5, for $i=1,2$.

to create even higher dimensional PL spheres, this is called the J-construction, see [14]. More precisely, for $J=(j_1,\dots,j_m)$ an m-uple of positive integers, we write $K(J)$ for the PL sphere having j_v copies of the vertex v of K after iterated wedge operations. PL spheres which are minimal with respect to this operation are called the *seeds*: they cannot be expressed as the wedge of another PL sphere. Therefore, every PL sphere can be obtained as a repeated wedge from a seed with the same Picard number. One major result of Choi and Park is that there are finitely many toric colorable seeds with a fixed Picard number [46]. More precisely, the number of vertices of a toric colorable seed with Picard number p is at most 2^p, the inequality being strict when $p \geqslant 3$.

There is a constructive way of generating characteristic maps over any $K(J)$, starting from the characteristic maps over K, called the *puzzle method*, introduced by Choi and Park in [46], and explained in what follows. It relies on an inductive step which from two characteristic maps λ_1 and λ_2 over K yields a characteristic map denoted by $\lambda_1 \wedge_v \lambda_2$ over $\mathrm{wed}_v(K)$, when it exists, and in that case they are called *v-adjacent*, see Chapter 6 for more details. A *wedge-stable set based on* K is a collection $\mathcal{X}$ of characteristic pairs (L,λ) such that:

— Every L is equal to $K(J)$ for some J,

— Whenever (L,λ_1) and (L,λ_2) are in $\mathcal{X}$, then $(\mathrm{wed}_v(L),\lambda_1\wedge_v\lambda_2)$ is also in $\mathcal{X}$, for every vertex v of L such that λ_1 and λ_2 are v-adjacent.

Let us define a *toric wedge induction* as in my joint work with Suyoung Choi and Hyeontae Jang [41]. Let $\mathcal{P}$ be a property on characteristic pairs. Let $\mathcal{X}$ be a wedge-stable set based on K. If we verify that:

— **Base case:** Every pair $(K,\lambda)\in\mathcal{X}$ satisfies $\mathcal{P}$.

— **Inductive step:** When $(L,\lambda_1),(L,\lambda_2)\in\mathcal{X}$ are v-adjacent and satisfy $\mathcal{P}$, then the characteristic pair $(\mathrm{wed}_v(L),\lambda_1\wedge_v\lambda_2)\in\mathcal{X}$ also satisfies $\mathcal{P}$.

Then, the property $\mathcal{P}$ is verified over all $\mathcal{X}$. Notably, Choi and Park used this method for showing the projectivity of certain toric manifolds in [44, 45]. In fact, in that case, a pair (L,λ) corresponds to the normal fan of Delzant polytope and a well-chosen polytopal wedge [98] yields the normal fan of Delzant polytope whose associated pair is $(\mathrm{wed}_v(L),\lambda\wedge_v\lambda)$.

Contributions In Chapter 5, we study the puzzle method in the mod 2 case and provide an algorithm based on it. As this algorithm is constructive, it happens to be more efficient than the branch-and-bound algorithm given by Garrison and Scott in [75], which was the most commonly used before for enumerating mod 2 characteristic maps over simplicial complexes. These results were published in 2022 in the Pacific journal of Mathematics [49].

In Chapter 6, we formalize different versions of the toric wedge induction and use it for solving the *toric lifting problem* for PL sphere with Picard number at most 4. This result uses the list of toric colorable seeds obtained in Chapter 4, which comes from a joint work with Suyoung Choi and Hyeontae Jang [41].

Outline

The rest of the thesis is organized as follows. We start with a preliminary Chapter 2 that serves as a basis containing all the cornerstone tools and objects appearing here. Then, we focus on combinatorial optimization with Chapter 3 which contains a submitted work with Patrick Chervet and Roland Grappe [35]. This is followed by three chapters on the subject of toric topology. Chapters 4 and 6 are based on joint works with Suyoung Choi and Hyeontae Jang, the first one published in Crelle's Journal [39], and the second one Published in the Journal of the London Mathematical Society [41]. Chapter 5 is based on a joint work with Suyoung Choi, published in the Pacific Journal of Mathematics [49]. Finally, the thesis ends with Chapter 7 that contains a quick overview of my ongoing and future works.

* *
*

2 Preliminaries

"This Weighted Companion Cube will accompany you through the test chamber. Please take care of it."

GLaDOS — Portal

In all that follows we consider the Euclidean vector space $\mathbb{R}^n$, for some $n \geqslant 1$. Its canonical basis is composed of the vectors $e_1, \dots, e_n$. It comes equipped with the usual inner product that we write $x^\top y = \sum_{i=1}^n x_i y_i$, for $x, y \in \mathbb{R}^n$, and $x^\top$ the transpose of x.

A vector is *integer* when it has integer entries. It is *primitive* when the gcd of its coordinates is 1. By a 0,1 vector, respectively a 0,$\pm$1 vector, we mean a vector whose coordinates are all in $\{0, 1\}$, respectively in $\{0, 1, -1\}$. The same definitions apply to points and to matrices.

Some definitions that have already been provided in the introduction are repeated here for the sake of completeness.

2.1 From linear Euclidean geometry to combinatorics and back

The main references for this section are the book of Buchstaber and Panov [26] for toric topology, the book of Gubeladze and Bruns [85] and the one of Ziegler [138] for polytopes, the book of Conforti, Cornuéjols, and Zambelli [53] for polyhedra and optimization, and finally the book of Oxley [112] for matroid theory.

This section is composed of four subsections. The first one introduces the different objects from linear Euclidean geometry and their connections. The second describes many combinatorial objects that will appear in this thesis. The third section describes hybrid objects that are both geometrical and combinatorial. The fourth one describes the key operation used in this thesis: the wedge operation.

Find in Figure 2.1 a diagram that gathers most of the objects that will appear in this thesis, together with their relations. This diagram shall accompany the reader in its journey, just as a map for the geometrico-combinatorial world.

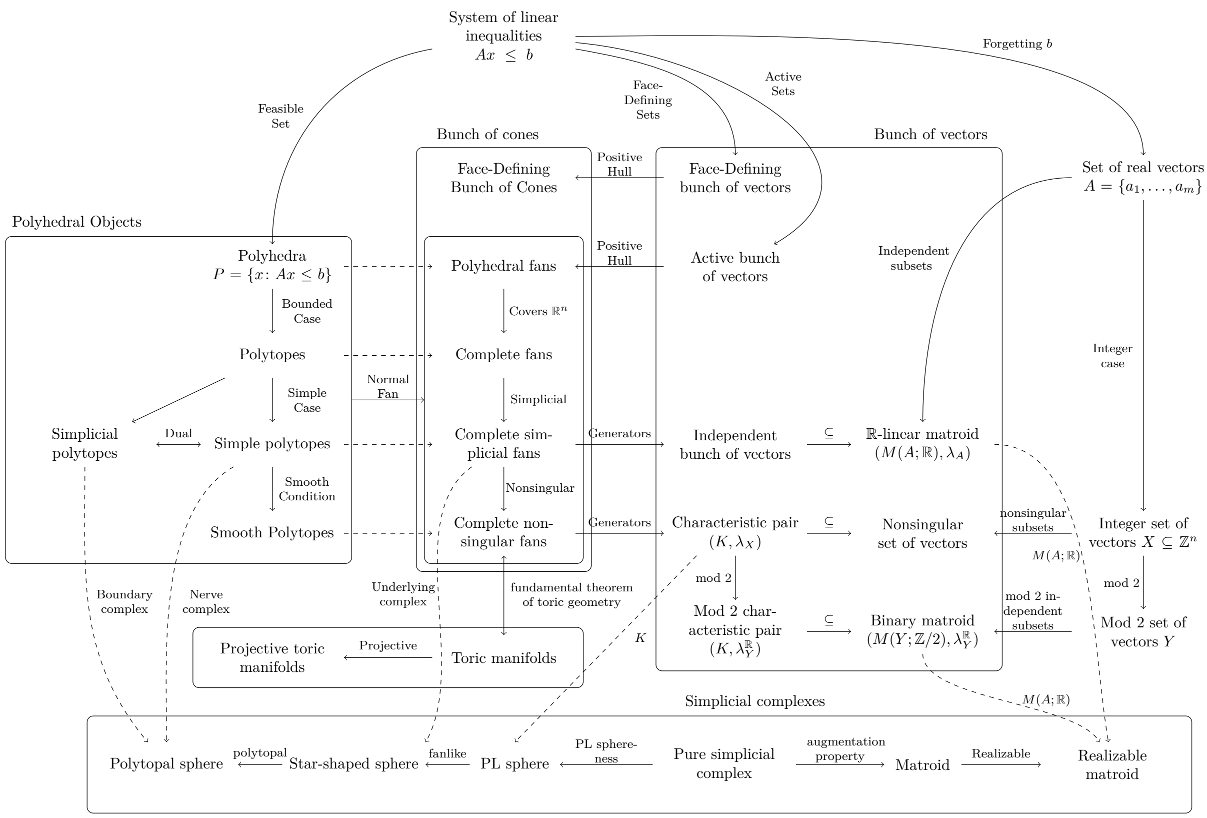


Figure 2.1 – Companion diagram illustrating the relationships between the key objects introduced in this chapter.

2.1.1 Objects from Euclidean geometry

System of linear inequalities. Let $a_1, \dots, a_m$ be some vectors of $\mathbb{R}^n$, and $b_1, \dots, b_m \in \mathbb{R}$ be scalars, then the associated *system of linear inequalities* is written as

$$\begin{cases} a_1^\top x \leqslant b_1 \\ a_2^\top x \leqslant b_2 \\ \quad \vdots \\ a_m^\top x \leqslant b_m \end{cases}, \text{ or more compactly, } Ax \leqslant b,$$

where x is an unknown point of $\mathbb{R}^n$, A is the $m \times n$ matrix whose rows are the $a_i^\top$'s, and b is the vector with entries b_i. A *solution* of this system is a point x satisfying all the inequalities, and its *feasible set* is the set of all its solutions: $P(A, b) = \{x \in \mathbb{R}^n : Ax \leqslant b\}$.

Polyhedra. Given a non-zero vector $\alpha \in \mathbb{R}^n$ and a scalar $\beta \in \mathbb{R}$, the *(affine) hyperplane* orthogonal to α at *height* β is

$$H_{\alpha,\beta} = \{x \in \mathbb{R}^n : \alpha^\top x = \beta\},$$

and is of dimension $n - 1$. It is *linear* whenever $\beta = 0$. A hyperplane splits the ambient space into two parts, called *half-spaces*, which are

$$H^{\leqslant}_{\alpha,\beta} = \{x \in \mathbb{R}^n : \alpha^\top x \leqslant \beta\} \text{ and } H^{\geqslant}_{\alpha,\beta}\{x \in \mathbb{R}^n : \alpha^\top x \geqslant \beta\}.$$

A *polyhedron* $P \subseteq \mathbb{R}^n$ is the intersection of finitely many half-spaces, which is furthermore described minimally [1]. More explicitly, it is expressed as

$$P = P(A,b) = \bigcap_{i=1}^{m} H^{\leqslant}_{a_i,b_i}, \text{ such that } \bigcap_{i=1, i\neq j}^{m} H^{\leqslant}_{a_i,b_i} \neq P \text{ for all } j = 1,\ldots,m, \tag{2.1.1}$$

with some vectors $a_1,\ldots,a_m \in \mathbb{R}^n$ and scalars $b_1,\ldots,b_m \in \mathbb{R}$. Therefore, it is equivalently defined as feasible set of a system of linear inequalities $Ax \leqslant b$, and in that case, P is called its *associated* polyhedron. The polyhedron P is *rational* when all the half-spaces defining P can be chosen by mean of rational vectors $a_i^\top$'s and rational b_i's. The *dimension* of a polyhedron P is the dimension of the affine space it spans, denoted by $\dim(P)$. In particular, P is *full-dimensional* when its dimension equals the dimension of its ambient space. A *face* of a polyhedron P is the intersection of P with any hyperplane such that one of its half-spaces contains P, and is again a polyhedron. A face of dimension k is called a *k-face*. Let $Ax \leqslant b$ be a system of linear inequalities and P its associated polyhedron. A subset of rows $\{a_i\}_{i\in I}$ of A is *face-defining* if it is linearly independent and there is a face of P whose affine hull is $\{x\colon a_i^\top x = b_i, i \in I\}$.

Polytopes. Let $p_1,\ldots,p_k$ be points of $\mathbb{R}^n$, their *convex hull* is

$$\operatorname{conv}(p_1,\ldots,p_k) = \left\{ \sum_{i=1}^{k} \alpha_i v_i \colon \sum_{i=1}^{k} \alpha_i = 1, \alpha_i \geqslant 0, i = 1,\ldots,k \right\}.$$

A *polytope* $P \subseteq \mathbb{R}^n$ is a minimally described convex hull of a finite set of points $\{v_1,\ldots,v_k\} \subseteq \mathbb{R}^n$, called its *vertex set*, and denoted by $V(P)$, as follows:

$$P = \operatorname{conv}(v_1,\ldots,v_k), \text{ such that } \operatorname{conv}(\{v_j : j \neq i\}) \neq P, \text{ for all } i = 1,\ldots,k.$$

In that case, we write P as $\operatorname{poly}(v_1,\ldots,v_k)$ instead of $\operatorname{conv}(v_1,\ldots,v_k)$ to insist it is a polytope.

Minkowski-Weil's theorem states that polytopes are bounded polyhedra. Therefore, in all that follows, we will consider polytopes either as the convex hull of their set of vertices $\operatorname{poly}(V)$ or as their minimal polyhedral representation $P(A,b)$, depending on the context. In particular, the vertices of P are the 0-faces, and for a face F of P, we have $V(F) \subseteq V(P)$. The faces of dimension 1 are the *edges*, those of dimension $\dim(P) - 1$ are the *facets*, and those of dimension $\dim(P) - 2$, the *ridges*. The *proper* faces of P are those of dimension strictly smaller than $\dim(P)$. Note that, when P is full-dimensional, since it is minimally described, its facets are one-to-one associated with the vectors $a_1,\ldots,a_m$. A polytope is *integer* if its vertices are integer, in particular it is rational.

The simplest polytope in $\mathbb{R}^n$ is the *canonical n-simplex* $\Delta^n = \operatorname{poly}(\mathbf{0}, e_1,\ldots,e_n)$. Any polytope which might be changed to Δ^n by a bijective affine transformation is called an *n-simplex*, or simply a *simplex*, see Figure 1.5.

A polytope is *simplicial* if all its proper faces are simplices. A polytope P is *simple* if all vertices of P have exactly $\dim(P)$ incident edges, or equivalently, are in exactly $\dim(P)$ facets. In the full-dimensional case, this is equivalent to requiring the vectors spanning these edges to have a nonzero determinant. A polytope P is *smooth*, *non-singular*, or *Delzant*, when at each vertex v of P, the set of primitive vectors generating the edges incident to v are part of a basis of $\mathbb{Z}^n$, and in the full-dimensional case, this is equivalent to their determinant being ± 1. Note that smooth polytopes are simple. In the combinatorial optimization literature, a smooth polytope $P = \{x\colon Ax \leqslant b\}$ corresponds to a *non-degenerate system* $Ax \leqslant b$.

1. The minimality assumption is not standard but practical in this thesis.

Cones. Let $r_1, \dots, r_k$ be vectors of $\mathbb{R}^n$, their *positive span* is

$$\mathrm{pos}(r_1, \dots, r_k) = \left\{ \sum_{i=1}^{k} \alpha_i r_i \colon \alpha_i \geqslant 0 \right\}.$$

A *(finitely generated) cone* $C \subseteq \mathbb{R}^n$ is a minimally described positive span of a finite set of vectors $G = \{g_1, \dots, g_k\}$, called its *generators* and denoted by $G(C)$, as follows:

$$C = \mathrm{pos}(g_1, \dots, g_k), \text{ such that } \mathrm{pos}(\{g_j \colon j \neq i\}) \neq C \text{ for all } i = 1, \dots, k.$$

In that case, we write C as $\mathrm{cone}(g_1, \dots, g_k)$ instead of $\mathrm{pos}(g_1, \dots, g_k)$ to insist the fact that C is a cone. Minkowski-Weil's theorem (for cones) states that cones are polyhedra defined by linear half spaces. Therefore, in all that follows, we will consider cones either as the positive hull of their set of generators $\mathrm{cone}(G)$ or as their minimal polyhedral representation $C(A) = \{x \in \mathbb{R}^n \colon Ax \geqslant 0\}$. A 1-dimensional cone is called a *ray*, and is generated by a single vector. The 1-faces of a cone are the rays generated by each of its generators. The *dual cone* to $C(A)$ is the cone spanned by the transpose of the rows of the matrix A, and is denoted by $C^\vee$.

A cone C is *pointed* when the set $\{x \in C \colon -x \in C\}$ contains only 0. A cone is *simplicial* if is generated by linearly independent vectors, in particular it is pointed. Moreover, when C is pointed and full-dimensional, its dual cone is also pointed and full-dimensional.

From polytopes to cones and back. From any pointed cone $C = \mathrm{cone}(G) \subseteq \mathbb{R}^n$ we can produce a polytope $P \subseteq \mathbb{R}^n$ of dimension $\dim(C) - 1$, as follows:

$$C \mapsto P = C \cap H \subseteq \mathbb{R}^n, \tag{2.1.2}$$

for $H \subseteq \mathbb{R}^n$ an affine hyperplane which intersects all the rays of C and that does not contain $\mathbf{0}$ in its interior. The generators of C are one-to-one associated with the vertices of P since they are the nonempty intersections of H with a ray of C. Therefore, C is generated by the vertices of P.

Conversely, from any polytope $P \subseteq \mathbb{R}^n$ of dimension at most $(n-1)$, and such that 0 is not in the affine span of P and such that $\mathrm{conv}(P \cup \{\mathbf{0}\})$ is of dimension n, we can produce a pointed cone by:

$$P \mapsto C = \mathrm{cone}\,(v : v \in V(P)) \subseteq \mathbb{R}^n. \tag{2.1.3}$$

Similarly, each ray of C is one-to-one associated with a vertex of P. The maps (2.1.2) and (2.1.3) provide a bridge between both worlds in the sense that each nonzero vector of C is associated to a point of P, and conversely.

The *cone over* P is a variant of (2.1.3). It is the cone generated by the points in $P \times \{1\} \subseteq \mathbb{R}^{n+1}$. Its generators are of the form $\begin{bmatrix} v \\ 1 \end{bmatrix}$ for $v \in V(P)$.

2.1.2 Combinatorics

In this thesis, combinatorial objects refer to collections of subsets of a finite set X. More precisely, let X be a finite set, the number of elements of X, or the *size* of X, is denoted by $|X|$. We will consider subsets of $\mathcal{P}(X) = 2^X = \{Y \subseteq X\}$, and for $k \in \mathbb{Z}_{\geqslant 0}$, we write $\binom{X}{k}$ for the set of elements in $\mathcal{P}(X)$ of size k.

Graphs. A *(simple loopless) graph* $G = (V, E)$ is a pair consisting of a finite set of vertices V and a set of edges $E \subseteq \binom{V}{2}$. An edge $\{v, w\}$ of G is often written as vw, for simplifying notations. A *path* in a graph is a sequence $(v_1, \dots, v_k)$ of vertices of G such that $v_i v_{i+1}$ is an edge of G, for every $i = 1, \dots, k-1$. A graph is *connected* if for every pair of vertices v, w of G, there exists a path $(v_1, \dots, v_k)$ in G with $v_1 = v$ and $v_k = w$.

Example 2.1.1. Let P be a polytope. The 1-*skeleton* of P is the graph $G = (V, E)$, where $V = V(P)$ and E is composed of the edges $\{v, w\}$ of P.

Hypergraphs. A *(simple loopless) hypergraph* $H = (V, E)$ is a pair consisting of a finite set of vertices V and a set of *hyperedges* $H \subseteq \bigcup_{k \geqslant 2} \binom{V}{k}$.

Example 2.1.2. Let P be a polytope with vertex set V. The k-*skeleton* of P is the hypergraph $G = (V, E)$ where E contains the sets of vertices $W \subseteq V$ such that $\mathrm{conv}(W)$ is a face of dimension at least 2 and at most k of P.

Example 2.1.3. Let P be a polyhedron whose facets are denoted by $F_1, \dots, F_m$. The *nerve complex* of P is the following set:

$$K_P = \{I \subseteq [m] \colon \cap_{i \in I} F_i \text{ is a non-empty face of } P\}.$$

And the pair $([m], K_P \setminus \{\{i\} \colon i \in [m]\})$ is a hypergraph.

Example 2.1.4. Let P be a polytope whose vertices are denoted by $v_1, \dots, v_m$. The *boundary complex* of P is the following set:

$$K^{\partial P} = \{I \subseteq [m] \colon \mathrm{conv}\,(v_i \colon i \in I) \text{ is a face of } P\}.$$

And the pair $([m], K^{\partial P} \setminus \{\{i\} \colon i \in [m]\})$ is a hypergraph.

Abstract simplicial complexes. An *(abstract) simplicial complex* on V is a set of *faces (or simplices)* $K \subseteq \mathcal{P}(V)$ such that:

(i) $\varnothing \in K$, and (contains the empty set)

(ii) $\forall \sigma \in K, \tau \subseteq \sigma \Rightarrow \tau \in K$. (stable by taking subfaces)

The *dimension* of K is $\dim(K) = \max_{\sigma \in K} |\sigma| - 1$, and K is *pure* when all its inclusionwise maximal faces have the same dimension. The elements of $\mathcal{P}(V) \setminus K$ are the *non-faces* of K, the set of non-faces of K is denoted by $\mathrm{NF}(K)$, and those which are minimal for inclusion are the *minimal non-faces*, or *missing faces*, of K, their set is denoted by $\mathrm{MNF}(K)$.

Example 2.1.5. When P is a full-dimensional simple polytope, then its nerve complex K_P is a pure simplicial complex.

Example 2.1.6. When P is a simplicial polytope, then its boundary complex $K^{\partial P}$ is a pure simplicial complex.

When a simplicial complex K is pure of dimension n, its *boundary complex* ∂K is the simplicial complex whose facets are the faces of dimension $n-1$ of K.

We define here some useful operations on simplicial complexes. Let K and L be two simplicial complexes on two disjoint vertex sets, V and W, respectively, and of dimension $n-1$ and $l-1$, respectively. The *join* of K and L is the simplicial complex

$$K * L = \{\sigma \cup \tau \colon \sigma \in K, \tau \in L\},$$

on $V \cup W$, and has dimension $n+l-1$. The join $K * \mu$, of K with $\mu = \{\varnothing, \{u\}\}$, for u a vertex not in V, is the *cone* over K. Let v be a vertex of K. The *deletion of* v in K is

$$K \backslash v = \{\sigma \in K \colon \sigma \not\ni v\},$$

the *link* of v in K is

$$\mathrm{lk}_K(v) = \{\sigma \in K \colon \sigma \not\ni v, \sigma \cup \{v\} \in K\},$$

and the *star* of v in K is

$$\mathrm{st}_K(v) = \{\sigma \in K \colon \sigma \cup \{v\} \in K\}.$$

Note that $\mathrm{st}_K(v) = \mathrm{lk}_K(v) * \mu$, for $\mu = \{\varnothing, \{v\}\}$.

Matroids. A *matroid* with *ground set* V is a simplicial complex $M \subseteq \mathcal{P}(V)$, which hence satisfies (i) and (ii), and furthermore satisfies the following additional property

(iii) $\forall I, J \in M, |I| < |J| \Rightarrow \exists v \in J \setminus I, I \cup \{v\} \in M.$ (augmentation property)

The faces of a matroid are referred to as the *independent sets* of M, and the inclusionwise maximal ones are the *bases*. Note that the augmentation property implies that all bases have the same cardinality, that is M is pure (as a simplicial complex). The elements $D \in \mathcal{P}(V) \setminus M$ are called the *dependent sets* of M. Inclusion-wise minimal dependent sets are the *circuits* of M. The *rank* in M of a subset $W \subset V$ is $\mathrm{rank}_M(W) = \max_{I \in M, I \subseteq W} |I|$. The rank of M is $\mathrm{rank}(M) = \mathrm{rank}_M(V)$. The *dual* of a matroid M is the matroid $M^* \subseteq \mathcal{P}(V)$ whose bases are the complementary in V of the bases of M. Its circuits are called the *cocircuits* of M, its bases the *cobases*.

2.1.3 Hybrid objects

Up to now, we have introduced geometrical objects which appear from vectors or points of $\mathbb{R}^n$ and abstract combinatorial objects separately. We now introduce hybrid objects that link both worlds. Namely, they appear as a pair consisting of an abstract combinatorial object on m elements, together with a set of m vectors or m points, and such that the pair satisfies some geometrical properties.

Hybrid objects associated to vectors

More precisely, let A be a finite collection [2] of m vectors [3] of $\mathbb{R}^n$, or equivalently the set of rows of an $m \times n$ matrix. As we systematically do, we write $A = \{a_1, \dots, a_m\}$, that is labeling the elements of A by integers in the set $[m]$. The map $\lambda_A \colon [m] \to \mathbb{R}^n$ defined by $\lambda_A(i) = a_i$ encodes this labeling of the set A, and each subset of vectors of A is identified to the associated subset of $[m]$ by $\lambda_A(I) = \{a_i\}_{i \in I}$, for all $I \in \mathcal{P}([m])$. Hence we can write $A = \lambda_A([m])$. Note that we may omit the subscript A when we directly consider a map $\lambda \colon [m] \to \mathbb{R}^n$.

In Chapters 4, 5, and 6, involving toric topology, we will always use this map λ in order to split the combinatorial structure from the geometrical one.

Bunch of vectors. Let $A = \{a_1, \dots, a_m\}$ be a finite collection of vectors of $\mathbb{R}^n$. A *bunch of vectors* based on A is a subset $\mathcal{B} \subseteq \mathcal{P}(A)$. Or equivalently, it is defined as a pair (X, λ), where $X \subseteq \mathcal{P}([m])$ and $\lambda \colon [m] \to \mathbb{R}^n$ maps i to the vector $\lambda(i) = a_i$. We say that a bunch of vector has some property $\mathcal{P}$ when all its elements satisfy $\mathcal{P}$. For instance, a bunch of vectors is linearly independent if all its sets of vectors are linearly independent.

2. By a collection, we mean that some elements may be repeated

3. Here, we consider the elements of $\mathbb{R}^n$ as vectors as they will be generators of cones.

To any solution x of a linear set of inequalities $Ax \leqslant b$ involving A, the *active set of rows* for x in $Ax \leqslant b$ is the set A_{I_x} consisting of all the inequalities of $Ax \leqslant b$ that reach equality at x, that is $A_i x = b_i$ if and only if $i \in I_x$. The *active bunch of vectors* associated with $Ax \leqslant b$ is the subset of $\mathcal{P}(A)$ containing the active sets A_{I_x}, for all solutions x. Namely, it is

$$\mathcal{B}(Ax \leqslant b) = \{A_{I_x} : x \in \mathbb{R}^n\}.$$

The *face-defining bunch of vectors* associated with $Ax \leqslant b$ is the subset of $\mathcal{P}(A)$ that is composed of every set of face-defining vectors in $Ax \leqslant b$. In particular, it is a linearly independent bunch of vectors.

More generally we define bunch of vectors in any space $\mathbb{F}^n$, for $\mathbb{F}$ some field. For instance, in this thesis, we will mainly consider the cases $\mathbb{F} = \mathbb{R}$, $\mathbb{F} = \mathbb{Z}$, $\mathbb{F} = \mathbb{Z}/2$, or $\mathbb{F} = \mathbb{Z}/3$.

The *($\mathbb{F}$-)linear matroid* associated with a finite collection of vectors $A \subseteq \mathbb{F}^n$ is the bunch of vectors containing every $\mathbb{F}$-independent subsets of vectors of A. If we write $A = \{a_1, \dots, a_m\}$, it is encoded as a pair (M, λ) where

$$M = M(A; \mathbb{F}) := \{I \subseteq [m] \colon \{a_i\}_{i \in I} \text{ are } \mathbb{F}\text{-linearly independent}\},$$

and $\lambda \colon [m] \to \mathbb{F}^n$ such that $\lambda(i) = a_i$. The simplicial complex $M(A; \mathbb{F})$ is called the *abstract ($\mathbb{F}$-)linear matroid* associated to A. When $\mathbb{F} = \mathbb{Z}/2$ it is called a *binary matroid*, and when $\mathbb{F} = \mathbb{Z}/3$, it is called a *ternary matroid*. A matroid is *realizable* when it can be described as the linear matroid associated to some collection of vectors.

When $\mathbb{F} = \mathbb{Z}$, a set of integer vectors is *non-singular* when it is part of a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. Hence, we say that a bunch of vectors is *non-singular* when it contains only non-singular sets of vectors. The *non-singular bunch of vector* associated with A is the set of all non-singular subsets of vectors of A. Note that this is a subset of the linear matroid associated with A.

Bunch of cones. A *bunch of cones* is a is a collection of polyhedral cones of $\mathbb{R}^n$.

From a bunch of vector $\mathcal{B}$ based on A, we naturally associate a bunch of cones (based on A), which is the set consisting of the non-negative hull of each set of vectors in the bunch of vectors. Namely, it is expressed as

$$\mathcal{C}(\mathcal{B}) = \{\mathrm{pos}(A') \colon A' \in \mathcal{B}\}.$$

Note that some vectors of A may become useless as they may not be generators of $\mathrm{pos}(A')$, and are hence forgotten in the description. More precisely, when $\mathcal{B} = (X, \lambda)$, its associated bunch of cones is the pair (X', λ) such that $\{\mathrm{pos}(\lambda(I)) \colon I \in X\} = \{\mathrm{cone}(\lambda(I)) \colon I \in X'\}$.

Conversely, from a bunch of cones $\mathcal{C}$, a bunch of vectors associated to $\mathcal{C}$ is obtained by choosing a set of generators for every cone in $\mathcal{C}$.

The face-defining bunch of cones of a linear system $Ax \leqslant b$ is the one associated to the face-defining bunch of vectors of P. It contains only simplicial cones.

Fans. A *fan* Σ based on A is a bunch of cones of $\mathbb{R}^n$ such that:

(i) every cone in Σ is expressed as $\mathrm{cone}(A')$ for some $A' \subseteq A$, (based on A)

(ii) every face of a cone in Σ is in Σ, and (stable by taking faces)

(iii) the intersection of any two cones in Σ is a face of both. (stable for intersections)

It is therefore combinatorially encoded by the pair (K, λ) with $\Sigma = \{\mathrm{cone}(\lambda(I)) \colon I \in K\}$ such that if $\sigma, \tau \in X$ then $\sigma \cap \tau \in K$.

A fan is *complete* when the union of its cones is the whole space $\mathbb{R}^n$, it is *simplicial* when its cones are simplicial. Furthermore, it is *rational* when all its cones are generated by rational, equivalently by integer vectors. Finally, it is *non-singular*, or *smooth*, when all its cones are non-singular, namely are all generated by a part of a basis of $\mathbb{Z}^n$.

When a linear system $Ax \leqslant b$ describes a polyhedron P, the bunch of cones associated to the active bunch of vectors of $Ax \leqslant b$ describes a fan, called the *normal fan* of the polyhedron, it is rational when P is rational. Note that the face-defining bunch of cones "covers" the normal fan of P, in the sense that every cone in the latter can be covered by the union of some cones from the first.

The normal fan of a polytope is complete, and is encoded by a pair (K, λ) where $K = K_P$ is the nerve complex of P. The normal fan of a simple polytope is simplicial, and it is non-singular if and only if the polytope is smooth. In that case, the associated active bunch of cones coincides with the normal fan.

Hybrid objects associated to points

Let $X = \{p_1, \ldots, p_m\}$ be a finite set of points[4] of $\mathbb{R}^n$. Similarly to the previous section, the map $\phi\colon [m] \to \mathbb{R}^n$, defined by $\phi(i) = p_i$ encodes X.[5] Any subset of $\mathcal{P}(X)$ is identified to the associated subset of $[m]$ by the correspondence $\phi(I) = \{p_i\}_{i\in I}$.

Let $\{e_i\}_{i\in[m]}$ be the canonical basis of $\mathbb{R}^m$. One way to represent $I \subseteq [m]$ is by the mean of its *characteristic vector* in $[m]$: $e_I = \sum_{i\in I} e_i$. In particular, I is seen as a 0,1 vector e_I whose coordinate at e_i is 1 if $i \in I$ and 0 if $i \in [m] \setminus I$. This is a very convenient and widely used way of representing combinatorial objects in some finite dimensional space.

The *characteristic polytope* of a subset $\mathcal{S} \subseteq \mathcal{P}([m])$ is

$$P_{\mathcal{S}} = \mathrm{poly}(e_I\colon I \in \mathcal{S}) \subseteq [0,1]^m.$$

In that case, the pair encoding this polytope is $(\mathcal{S}, \phi)$, with $\phi(i) = e_i$.

Example 2.1.7. Let $G = (V, E)$ be a graph. A *stable set* of G is a subset of vertices $S \subseteq V$ such that no pair of vertices in S is an edge of G. The stable set polytope of a graph $G = (V, E)$ is the characteristic polytope of the subset $\mathcal{S} \subseteq \mathcal{P}(V)$ that contains every stable sets of G. The stable set polytope of a graph with no edges is the hypercube $[0,1]^V$. The one of the complete graph is the standard simplex $\mathrm{poly}(\mathbf{0}, e_1, \ldots, e_{|V|})$.

General piecewise linear geometry. In piecewise-linear (PL) geometry, we consider objects of $\mathbb{R}^n$ that look like smooth geometrical objects, but which are assembled by gluing together simplices of $\mathbb{R}^n$, so they are piecewise-linear. Triangulations of smooth compact algebraic varieties are examples, where an algebraic variety of $\mathbb{R}^n$, is the zero locus in $\mathbb{R}^n$ of a set of polynomials in n variables. We refer the reader to [118, 58, 59] for more details about PL geometry.

Example 2.1.8. The algebraic variety defined by the zero locus of the polynomial in two variables $x^2 + y^2 - 1$ in $\mathbb{R}^2$ is the unit circle $S^1 = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 = 1\}$.

A *geometric simplicial complex* $|K| \subset \mathbb{R}^n$ is encoded by a pair (K, ϕ) composed of a simplicial complex K on $[m]$ together with a map $\phi\colon [m] \to \mathbb{R}^n$ such that $|K| = \bigcup_{I\in K} \mathrm{conv}(\phi(i)\colon i \in I)$, each element of the union is called a *geometric simplex* of $|K|$. In that case, we say that $|K|$ is a *geometric realization* of K. A geometrical simplicial complex $|K|$ is a topological object, its open sets are the intersection of the open sets of $\mathbb{R}^n$ with $|K|$. Therefore, if $|K|$ is not self-intersecting, that is there are two simplices of $|K|$ which intersect in their relative interiors, we can give an atlas on it, namely a set of open sets of $|K|$ together with piecewise linear maps from these open

4. Here, we consider the elements of $\mathbb{R}^n$ as points as they will describe the vertices of convex hulls.
5. We use a different notation for this map, ϕ instead of λ, to distinguish the way we use them afterwards.

sets to open sets of $\mathbb{R}^{\dim(K)}$ which satisfy compatibility and restriction conditions, and it has a *PL structure* when one can pass from two open sets of an atlas by mean of piecewise linear functions.

Example 2.1.9. Drawing in $\mathbb{R}^2$ any convex polygon yields, up to a smooth deformation, a triangulation of the circle S^1 and is therefore a PL manifold, which is furthermore a PL sphere, as defined below.

Let K and L be two PL manifolds. Then L is a *subdivision* of K when there exists two realizations $|K|$ of K and $|L|$ of L such that every geometric simplex of $|L|$ is contained in a geometric simplex of $|K|$, and every geometric simplex of $|K|$ is the union of geometric simplices of $|L|$. Two PL manifolds are *PL homeomorphic* when they share a common subdivision.

Combinatorial piecewise linear geometry. We now provide more combinatorial conditions for K to be a PL manifold as well as definitions of other type of combinatorial manifolds. Note that first, since we need an atlas, which is a collection of PL-homeomorphisms from open sets of K to open sets of $\mathbb{R}^d$, when $d = \dim K$, PL manifolds need to be pure.

Let K be a pure simplicial complex with vertex set $[m]$ and of dimension $n-1$, namely its top dimensional faces, the *facets* of K, are of size n. The *Picard number* of K is $\operatorname{Pic}(K) = m-n$. The faces of size $n-1$ are called the *ridges*. A simplicial complex is a *weak pseudo-manifold* if it is pure and every ridge is contained in exactly two facets. Additionally, it is a *pseudo-manifold* (without boundaries) if its ridge-facet graph is connected. One example of pseudo-manifolds is the boundary of the simplex, denoted by $\partial\Delta^n$, whose facets are the subsets of size n of $[n+1]$, and which has Picard number 1. Any set of affinely independent points $v_1, \dots, v_{n+1} \in \mathbb{R}^n$ yields a geometric realization $|\partial\Delta^n|$ of $\partial\Delta^n$ that is piecewise-linearly homeomorphic to the sphere S^{n-1}. A simplicial complex K of dimension $n-1$ is a *PL sphere* if there exists a subdivision of K and a subdivision of $\partial\Delta^n$ such that these subdivisions are isomorphic. It is a *PL manifold* if the link of each of its faces is a PL sphere. A *polytopal sphere* is the boundary complex of a simplicial polytope, or equivalently the nerve complex to a simple polytope. We have the following hierarchy on pure simplicial complexes:

$$\text{polytopal spheres} \quad \subseteq \quad \text{PL spheres} \quad \subseteq \quad \text{pseudo-manifolds} \quad \subseteq \quad \text{weak pseudo-manifolds.}$$

Hybrid objects in toric topology

The fundamental theorem of toric geometry states that smooth complete toric varieties, called *toric manifolds* here, are one-to-one associated with smooth complete rational fans. From this equivalence, in all that follows, we only consider the second viewpoint on fans and will not define what toric varieties are in the context of geometry, for which good references are [108], [73], and [57].

Motivated by this, one research direction in toric geometry and more generally in toric topology is to try to characterize every such smooth complete rational fans. Since it is difficult to study smooth complete rational fans, many new hybrid objects that are easier to study have appeared in toric topology. These objects are obtained after dropping a few conditions in the definition of a complete non-singular rational fan. Let $\mathcal{C} = (K, \lambda)$ be a bunch of cones, then, by [44] it is a complete non-singular fan of $\mathbb{R}^n$ if and only if:

1. K is a pure simplicial complex which is a PL sphere, (complete fan)
2. for every $i \in [m]$, $\lambda(i) \in \mathbb{Z}^n$ that is $\lambda\colon [m] \to \mathbb{Z}^n$, (rational)
3. for every face $I \in K$, the set of vectors $\lambda(I)$ is non-singular, (non-singularity condition)
4. for every point of $\mathbb{R}^n$, there is a unique face I of K such that x is in the relative interior of $\operatorname{cone}(\lambda(I))$. (complete fan)

Characteristic maps and characteristic pairs. Let K be a PL sphere with vertex set $[m]$ and of dimension $n-1$. A *characteristic map* over K is a map $\lambda\colon [m] \to \mathbb{Z}^n$, satisfying the *non-singularity condition* for K: for every face $I \in K$, the set of vectors $\lambda(I)$ is non-singular, that is, they form a part of a basis of $\mathbb{Z}^n$. In that case, the pair (K, λ) is called a *characteristic pair*, and it is simply a non-singular bunch of vectors that is stable by taking subsets. When K admits a characteristic map, we say that it is *toric colorable*. We say that a characteristic pair is *fan-giving* when its associated bunch of cones is a complete non-singular fan. In that case, it yields a toric manifold, and K is a PL sphere that is called *fanlike*. We say a fan-giving characteristic pair is *projective* when is associated fan is the normal fan of a smooth polytope. In that case, the associated toric variety is projective, meaning it can be embedded as a subvariety of the complex projective space $\mathbb{CP}^N$, for some positive integer N.

Remark 2.1.10. Note that the non-singularity condition on K is equivalent to the the fact that K is a subset of L where (L, λ) is the non-singular bunch of vectors associated to $\lambda([m])$.

A *mod* 2 *characteristic map* over K is a map $\lambda^{\mathbb{R}}\colon [m] \to \mathbb{Z}_2^n$, satisfying the *mod* 2 *non-singularity condition* for K: for every face $I \in K$, the set of vectors $\lambda^{\mathbb{R}}(I)$ is mod 2 independent. In that case, the pair $(K, \lambda^{\mathbb{R}})$ is called a *mod* 2 *characteristic pair*. Note that by the non-singularity condition, the mod 2 reduction of every characteristic map over K is a mod 2 characteristic map over K. When K admits a mod 2 characteristic map, we say that it is $\mathbb{Z}_2^n$*-colorable*. Considering mod 2 characteristic map is very convenient as, in contrast with characteristic maps, there is a finite number of them when we fix n and m.

Remark 2.1.11. Note that the mod 2 non-singularity condition on K is equivalent to the fact that K is a subset of $M(\lambda^{\mathbb{R}}; \mathbb{Z}/2)$, the binary matroid associated to $\lambda^{\mathbb{R}}$.

2.1.4 Wedge operation

We conclude this section by introducing the wedge operation, which plays a central role in this thesis.

Polytopal wedge operation [98]

In what follows, for any vector v of $\mathbb{R}^n$ and scalar β of $\mathbb{R}$, we denote by (v, β) the vector of $\mathbb{R}^{n+1}$ whose first n entries coincide with those of v and whose $(n+1)$-st entry is β. We say that two polytopes are *combinatorially equivalent* when their face lattices are isomorphic, that is they have isomorphic nerve complexes.

Let P be a polytope of dimension n of $\mathbb{R}^n$, Σ_P its normal fan, and F a face of P. A *polytopal wedge* of P at F is a polytope that is combinatorially equivalent to the polytope $\hat{P}$ constructed as follows.

1. Choose a hyperplane $H_{a_F,b_F} = \{x \in \mathbb{R}^n \colon a_F^\top x = b_F\}$, with a_F a vector of $\mathbb{R}^n$ and b_F a scalar, such that $F = H \cap P$. In particular, a_F belongs to the relative interior of the cone normal to F in the normal fan of P.
2. Embed P into of $\mathbb{R}^{n+1}$ as $P' = P \times \{0\}$, its normal fan is

$$\begin{aligned}\Sigma_{P'} &= \{C \times \mathbb{R}\}_{C \in \Sigma_P} \\ &= \{\mathrm{cone}(\{(a, 0)\colon a \in G(C)\} \cup \{e_{n+1}, -e_{n+1}\})\}_{C \in \Sigma_P},\end{aligned}$$

   since it is obtained by embedding the generators of the cones into $\mathbb{R}^{n+1}$ by setting the coordinate $n+1$ to 0, and two new inequalities appear: $x_{n+1} \leqslant 0$ and $-x_{n+1} \leqslant 0$.

3. Consider the polyhedron $P^+ = P \times \mathbb{R}^+$, whose normal fan is therefore

$$\Sigma_{P+} = \{\text{cone}(\{(a,0)\colon a \in G(C)\}, -e_{n+1})\}_{C\in\Sigma_P},$$

since we have removed the inequality $x_{n+1} \leqslant 0$.

4. We define $\hat{P} = P^+ \cap H^{\leqslant}_{(a_F,1),b_F}$. Its normal fan is:

$$\Sigma_{P+} = \{\text{cone}(\{(a,0)\colon a \in G(C)\}, -e_{n+1})\}_{C\in\Sigma_P} \cup \{\text{cone}(\{(a,0)\colon a \in G(C)\}, (a_F,1))\}_{C\in\Sigma_P}.$$

and $\hat{P}$ is a polytopal wedge of P at F.

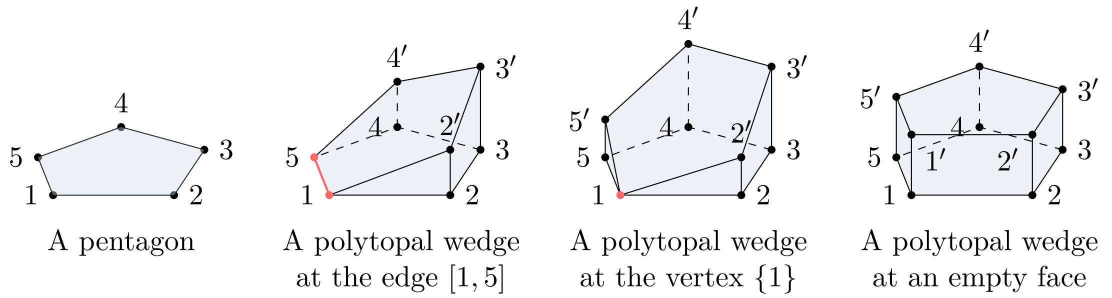


Figure 2.2 – Illustration of the polytopal wedge for different faces of a pentagon.

Remark 2.1.12. One can also picture the polytopal wedge of P at F as follows. Consider the polytopal wedge at an empty face $P \times [0,1]$ and smoothly squeeze the face $F \times [0,1]$ until it becomes $F \times \{0\}$, while keeping the structure of the other faces intact.

We have the following direct properties on polytopal wedges.

Proposition 2.1.13. *Let $\hat{P}$ be a polytopal wedge of P at a face F. Then,*

— $\dim(\hat{P}) = \dim(P) + 1$.

— *There exist two non co-linear affine hyperplanes H_1 and H_2 such that $H_1 \cap \hat{P}$ and $H_2 \cap P$ are combinatorially equivalent to P and $H_1 \cap H_2 \cap \hat{P}$ is combinatorially equivalent to F.*

Simplicial wedge operation [115]

Let K be a simplicial complex and v be a vertex of K. The *(simplicial) wedge* of K at v is the simplicial complex

$$\text{wed}_v(K) = (K\backslash v * \partial I) \cup (\text{lk}_K(v) * I),$$

on $(V \setminus \{v\}) \cup \{v_1, v_2\}$, for $I = \{\varnothing, \{v_1\}, \{v_2\}, \{v_1, v_2\}\}$ the full simplicial complex on $\{v_1, v_2\}$, two new vertices, and $\partial I = \{\varnothing, \{v_1\}, \{v_2\}\}$ its boundary, consisting of the two disjoint vertices v_1 and v_2, see Figure 2.3.

The polytopal wedge and the simplicial wedge operations are linked to each other in the case of simplicial polytopes as follows.

Proposition 2.1.14 ([115])**.** *Let P be a simple polytope and F a facet of P, then we have:*

$$K_{\hat{P}} = \text{wed}_v(K_P),$$

for v the vertex of K_P corresponding to F.

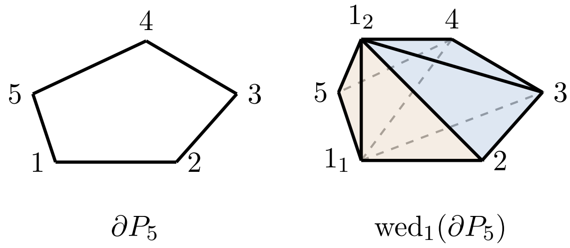


Figure 2.3 – Wedge at the vertex 1 of the boundary of a pentagon. The facets (triangles) corresponding to $\partial P_5 \setminus 1 * \partial I$ are in light indigo and the ones corresponding to $\mathrm{lk}_{\partial P_5}(1) * I$ are in light orange.

2.2 Optimization

Now that we have all the previously introduced tools in mind, we are well-equipped to introduce optimization problems. This section is inspired from different chapters (Chapter 3 and Chapter 4) of the excellent book of Conforti, Cornuéjols, and Zambelli [53].

In what follows, for every matrix A, we denote by A_i its row indexed by i, and by A_i^j its coefficient (i, j). Moreover, for every row index subsets I, we denote by A_I the submatrix of A having the corresponding rows.

2.2.1 Linear programming

Primal and dual problem, Fourier elimination. A *linear optimization problem* is the data of a system of linear inequalities $Ax \leqslant b$, for A an $m \times n$ matrix and b a point of $\mathbb{R}^m$, coming with a linear function $x \mapsto c^\top x$, for c some vector of $\mathbb{R}^n$. It aims to find a solution x^* satisfying $Ax \leqslant b$ that reaches the maximum, for the function $x \mapsto c^\top x$. We write this problem, called the *primal*, as follows:

$$\max\{c^\top x \colon Ax \leqslant b\}. \tag{2.2.1}$$

A useful method for checking the feasibility of a problem is the so-called *Fourier elimination*, which helps getting rid of a variable. Let $Ax \leqslant b$ be a system of linear inequalities. We aim to write an equivalent system to $Ax \leqslant b$ with one less variable, says x_n. We split the rows of A into three subsets: I^0 are the ones having a 0 coefficient at coordinate n, I^+ a positive coefficient, and I^- a negative one. For every row A_i of A at index $i \in I^+ \cup I^-$, divide the inequality $A_i x \leqslant b_i$ by the positive scalar $|A_i^n|$ to obtain $A'_i x \leqslant b'_i$. We obtain a new system $A'x \leqslant b'$, equivalent to A as follows:

$$\begin{array}{llll} \sum_{k=1}^{n-1} (A')_i^k x_k & +x_n & \leqslant b'_i, & i \in I^+ \\ \sum_{k=1}^{n-1} (A')_i^k x_k & -x_n & \leqslant b'_i, & i \in I^- \\ \sum_{k=1}^{n-1} A_i^k x_k & & \leqslant b_i, & i \in I^0. \end{array} \tag{2.2.2}$$

For each pair $i \in I^+$ and $j \in I^-$ we sum the two associated inequalities and add it as a row of (2.2.2). Finally, we remove the rows corresponding to I^+ and I^-, to finally obtain:

$$\begin{array}{rll} \sum_{k=1}^{n-1} ((A')_i^k + (A')_j^k) x_k & \leqslant b'_i + b'_j, & i \in I^+, j \in I^- \\ \sum_{k=1}^{n-1} A_i^k x_k & \leqslant b_i, & i \in I^0. \end{array} \tag{2.2.3}$$

If $(x_1, \ldots, x_n)$ satisfies $Ax \leqslant b$ then $(x_1, \ldots, x_{n-1})$ satisfies (2.2.3). In fact, the converse also holds.

Theorem 2.2.1 (Fourier elimination [53, Theorem 3.1]). *A point* $(x_1, \ldots, x_{n-1})$ *satisfies 2.2.3 if and only if there exists a point* $(x_1, \ldots, x_{n-1}, x_n)$ *which satisfies* $Ax \leqslant b$.

Note that this process can be repeated until there is no variables left. We eventually obtain the system $A^\circ x \leqslant b^\circ$ of the form $\mathbf{0} \leqslant b^\circ$, and $Ax \leqslant b$ has a solution if and only if b° has only non-negative entries. Note that every inequality of the system (2.2.3) is obtained as a non-negative combination of the inequalities of A.

Farkas' Lemma and the strong duality theorem. The *dual problem* to (2.2.1) is

$$\min\{b^\top y \colon A^\top y = c, y \geqslant \mathbf{0}\}. \tag{2.2.4}$$

We introduce *Farkas' Lemma* which gives an alternative to the existence of a solution to a system of linear inequalities, it links the dual problem to the primal one when $c = \mathbf{0}$.

Lemma 2.2.2 (Farkas' Lemma [53, Theorem 3.4]). *A system of linear inequalities* $Ax \leqslant b$ *is infeasible if and only if the system* $A^\top y = \mathbf{0}, b^\top y < 0, y \geqslant \mathbf{0}$ *is feasible.*

Theorem 2.2.3 (Strong duality Theorem and linear programming duality [53]). *The primal problem* (2.2.1) *has an optimal solution* x^* *if and only if its dual* (2.2.4) *has an optimal solution* y^**, and in that case we obtain the* linear programming duality*:*

$$\max\{c^\top x \colon Ax \leqslant b\} = \min\{b^\top y \colon A^\top y = c, y \geqslant \mathbf{0}\}.$$

Theorem 2.2.4 (Complementary slackness condition [53]). *Let* x *and* y *be feasible solutions to the primal problem* (2.2.1) *and to the dual problem* (2.2.4)*, respectively. Then* x *and* y *are optimal if and only if for every constraint indexed by* i*, we have* $y_i^\top(b_i - A_i x) = 0$.

From this, one should notice the following direct consequence:

Corollary 2.2.5. *Let* P *be the polyhedron associated to* $Ax \leqslant b$ *and let* F *be a face of* P*. Let* c *be a vector of* $\mathbb{R}^n$*. Then, an optimal solution for the maximization problem* $\max\{c^\top x \colon Ax \leqslant b\}$ *lies in* F *if and only if* c *lies in the cone spanned the active rows for* F*. Moreover, a solution* y^* *of* $A^\top y = c$ *is optimum only if the rows of* A *that correspond to its support are all active for* F*.*

Proof. Let A_I be the the set of active rows for F We first prove the "only if" direction. Assume that an optimal solution x^* lies in the relative interior of F, that is $a_i x^* = b_i \Leftrightarrow i \in I$. By the strong duality theorem, there exists an optimal solution y^* to the dual problem. and by the complementary slackness condition, y^* we have $y_i^*(b_i - A_i x^*) = 0$, for every i. In particular, when $i \notin I$ we must have $y_i^* = 0$, since $y^* \geqslant 0$. Hence, $c = A^\top y^*$ is in the cone spanned by the active rows for F. This also proves the second statement.

We now prove the "if" direction. Assume that $c^\top = \sum_{i \in I} \lambda_i A_i$, for $\lambda_i \geqslant 0, i \in I$. Let $x \in P$ and $x^* \in F$. Since $x^* \in F$, $\forall i \in I, A_i x^* = b_i$, $A_i x \leqslant b_i \Leftrightarrow b_i - A_i x \geqslant 0$, and $\lambda_i \geqslant 0, i \in I$, we have

$$c^\top x^* = \sum_{i \in I} \lambda_i A_i x^* = \sum_{i \in I} \lambda_i b_i \geqslant \sum_{i \in I} \lambda_i A_i x = c^\top x.$$

And hence for any point $x^* \in F$ and every point x of P, we have $c^\top x \leqslant c^\top x^*$. That is x^* is optimal. □

Corollary 2.2.5 tells us that in linear programming, the vectors in the normal cone to a face F of P are the objective functions which are optimal in F. In particular, when we can provide an explicit combinatorial description of the normal fan of a polyhedron, then it provides useful information for solving any linear program on this polyhedra. That is why in this thesis, we put a great importance to the study of normal fans of polyhedra, and more specifically of polytopes.

The simplex method. For now, we assume that our polyhedra are bounded, that is, they are polytopes. In practice, one of the most efficient methods for finding an optimum in a linear optimization problem is the so-called *simplex method* introduced by Dantzig in 1947. Intuitively, it can be described as follows. Let $P = \{x \colon Ax \leqslant b\}$ be a polytope and c a vector, both lying in $\mathbb{R}^n$.

1. Find a maximum number of linearly independent rows of A that reach equality in $Ax \leqslant b$, say $A_I x = b_I$, hence representing a vertex v of P, with a certain objective value $c^\top v$,
2. Choose a *pivot*, that is, find another row A_j of A such that replacing a row of A_I with A_j yields another vertex v' of P, which strictly increase the objective value, that is, $c^\top v' > c^\top v$.
3. Repeat 2. until no new row is found, and in that case that means an optimal vertex has been found.

Note that this is not how the simplex method is usually described and implemented. Abstractly, let G be the 1-skeleton of P, then the simplex method starts at some vertex of G and at each steps chooses an edge of G leading to another vertex which increases the previous objective value. Equivalently, we start from a maximal cone in the normal fan Σ of P and try to successively "cross walls" between the maximal cones of Σ until finding the maximal cone of Σ in which c lies within, see Corollary 2.2.5.

2.2.2 Combinatorial and discrete optimization

An *integer linear optimization problem* is given by taking a linear problem as in (2.2.1) to which we add *integrality constraints* to the entries of x. These integrality constraints encode the fact that we are looking at a discrete set of objects which are encoded as the integer points inside the feasible domain of the system of linear inequalities $Ax \leqslant b$. Most of the time, we identify discrete objects with 0,1 vectors by the mean of their characteristic vectors, see Section 2.1.3.

A major problem appears: since the vertices of the polytope associated to a bounded system of inequalities may not be integer, the simplex method is of no use: we may end up at a noninteger vertex that may be "far away" from an integer solution to the problem.

Totally dual integral systems. We say a system $Ax \leqslant b$ is *totally dual integral* (*TDI*) if the minimization problem in the linear programming duality

$$\max\{c^\top x \colon Ax \leqslant b\} = \min\{b^\top y \colon A^\top y = c, y \geqslant \mathbf{0}\}$$

admits an integer optimal solution for each integer vector $c^\top$ such that the maximum is finite. Recall that a *Hilbert basis* for a cone C is a minimal set of integer vectors whose non-negative integer combinations generate all the integer points of C. Hence, the following gives an equivalent formulation of total dual integrality, which furthermore motivates the study of the Hilbert basis of cones.

Theorem 2.2.6. *A system $Ax \leqslant b$ is totally dual integral if and only if every set A_I in the bunch of vectors $\mathcal{B}(Ax \leqslant b)$ forms a Hilbert basis for the cone* $\mathrm{pos}(A_I)$.

Proof. We first prove the "if" part. Let c be an integer vector such that the maximum is finite, and let x^* be an optimum, lying in some face F of P. By Corollary 2.2.5, c is in the cone associated to F in the normal fan of P. By assumption, the rows of A associated to F form a Hilbert basis for this cone. Hence, since c is an integer vector, it can be expressed as a non-negative integer combination of this Hilbert basis, that is $c = A^\top y$ for $y \in \mathbb{Z}^m_{\geqslant 0}$. By Corollary 2.2.5, this yields an optimal solution for the minimization problem.

For the “only if” part, let F be a face such that the active rows of A for F do not yield a Hilbert basis for the cone they generate. Hence, there is an integer vector c in this cone that cannot be expressed as an integer combination of the rows of A associated to F. Hence, either the whole system $A^\top y = c$ has no solution, a contradiction. Or, we need additional rows of A in order to express c, but this contradicts the complementary slackness condition and hence does not yield an optimal solution. □

Proving that a system is TDI is of major importance as it guarantees integrality of the underlying polyhedron when b is integer, and hence implies we can use the simplex method to the optimization problem.

Theorem 2.2.7 ([53, Theorem 4.26])**.** *Let $Ax \leqslant b$ be a totally dual integral system and b an integral vector. Then $P = \{x\colon Ax \leqslant b\}$ is an integral polyhedron.*

Interestingly, every integer polyhedron can be represented by a TDI system $Ax \leqslant b$, where A and b have integral entries.

Theorem 2.2.8 ([53, Theorem 4.27])**.** *If P is a rational polyhedron, there exists a TDI system $Ax \leqslant b$, where A has integral entries, such that $P = \{x\colon Ax \leqslant b\}$. Moreover if P is integer, then b can be chosen integer.*

The proof of Theorem 2.2.8 relies on adding the Hilbert basis elements of every cone in the normal fan of P to the system of inequalities $Ax \leqslant b$, and choosing the appropriate inequality $h^\top x \leqslant b_h$, for every Hilbert basis element h. This is one of the reasons why studying the Hilbert basis of rational cones is of major importance in combinatorial optimization.

Remark for what follows. Note that the following chapters are independent, and hence, some definitions and notations may change from one to another.

* *
*

3 Totally equimodular matrices: decomposition and triangulation

"Put your right hand in the box.
-What's in the box?
-Pain."

Reverend Mother Gaius Helen Mohiam — Dune

In this chapter, based on a submitted work with Patrick Chervet and Roland Grappe [35], we are motivated by the study of box-TDI systems and polyhedra. We study the class of matrices that appear in this context: totally equimodular matrices. Motivated by a polyhedral approach, we also study the cones generated by the rows of such matrices.

3.1 Introduction

We first recall the definitions of the involved classes of matrices. A full row rank matrix A of size $m \times n$ is *unimodular* when all its $m \times m$ nonzero subdeterminants are ± 1. An integer matrix A is *totally unimodular* if every row-induced submatrices of A that is of full row rank is unimodular. Note that this is equivalent to the usual definition which is that every non-singular square submatrix of A has determinant ± 1.

A full row rank matrix A of size $m \times n$ is *equimodular* when all its $m \times m$ nonzero subdeterminants have the same absolute value. In that case this value is called the *equideterminant* of A, and is denoted by $\operatorname{eqdet}(A)$. A matrix A is *totally equimodular* if all row-induced submatrices of full row rank of A are equimodular. Note that totally unimodular matrices are totally equimodular with all equideterminants equal to 1.

We have the following characterization of box-TDI systems.

Theorem 3.1.1 ([34])**.** *A system $Ax \leqslant b$ is box-TDI if and only if every face-defining submatrix of A is equimodular. In other words, $Ax \leqslant b$ is box-TDI if and only if its face-defining bunch of vectors is equimodular, that is, every set of vectors in this bunch of vectors is equimodular.*

Hence, the study of the cones appearing in the face-defining bunch of cones associated to polyhedra underlying box-TDI systems is fundamental. In this chapter, we study the specific case of totally equimodular cones, which are the cones generated by the rows of a full row rank totally equimodular matrix. This lays the first stone in the big project of understanding box-TDI systems from a geometrico-combinatorial perspective. In fact, this covers many cases such as simple box-TDI polytopes, since their normal fan is the same as their face-defining bunch of vectors. Moreover, by Theorem 3.1.1 since the face-defining bunch of vectors is linearly independent, all the cones involved are simplicial and equimodular and hence they are totally equimodular cones.

We believe that this general approach can help understanding better box-TDI systems and polyhedra.

Outline. We start by some preliminaries in Section 3.2. In Section 3.3, we introduce a motivating example for our study: simplicial cones generated by the characteristic vectors of the edges of a specific type of graphs. Then, Section 3.4 introduces some more definitions and notations used throughout.

The rest of the chapter is divided into two blocks. The first block (Sections 3.5 and 3.6) contains the main results with only sketches of proofs, while the second block (Sections 3.7 and 3.8) contains the detailed proofs. In Section 3.5, we first establish two new parallels between totally unimodular and totally equimodular matrices. We then present our decomposition theorem for totally equimodular matrices of full row rank. We also discuss connections with known matrix classes and propose a conjecture to refine the decomposition. Section 3.6 is divided in two. First, we provide the Hilbert basis of all te-cones. Then, we construct a regular unimodular Hilbert triangulation for most te-cones.

The second part starts with Section 3.7, which is devoted to the proof of the decomposition theorem. In Section 3.8, we prove that the aforementioned triangulation is indeed regular, unimodular, and Hilbert.

For instance, the following new result is a special case of our main triangulation theorem:

Corollary 3.1.2. *Simplicial box-totally dual integral cones in the non-negative orthant have the integer Carathéodory property.*

3.2 Preliminaries

3.2.1 Definitions.

A (finitely generated) *cone* C is the set of all non-negative linear combinations of a finite collection of vectors A:

$$C = \operatorname{cone}(A) = \left\{ \sum_{a \in A} \lambda_a a \colon \lambda_a \geqslant 0 \right\}.$$

Let $C \subseteq \mathbb{R}^n$ be a cone and A its set of *generators*, that is, A is inclusionwise minimal such that $C = \operatorname{cone}(A)$. Throughout, we suppose all cones to be *simplicial*, that is, their generators are linearly independent. Thus, they are *pointed*, that is, they contain no lines. The *dimension* $\dim(C)$ of C is the dimension of the linear subspace it spans. The cone C is *full-dimensional* if $\dim(C) = n$. A *face* of a cone C is the intersection of C with a hyperplane such that one of the half-spaces defined by this hyperplane contains C. As a result, any face of C is a cone generated by a subset of the generators of C. The *Minkowski sum* of two cones $\operatorname{cone}(A)$ and $\operatorname{cone}(B)$ is $\operatorname{cone}(A) + \operatorname{cone}(B) = \operatorname{pos}(A \cup B)$ and equals to $\operatorname{cone}(D)$ for some $D \subseteq A \cup B$.

An integer vector $x \in \mathbb{Z}^n$ is called *primitive* if the greatest common divisor of all its coefficients is 1. We will always assume the generators to be primitive. An integer vector h of C is a *Hilbert basis element* if we cannot express it as the sum of two nonzero integer vectors of C. A *Hilbert basis* $\mathcal{H} \subseteq C \cap \mathbb{Z}^n$ is a finite set of Hilbert basis elements generating all the integer points of the cone with non-negative integer coefficients. Here, since the cones are pointed, the Hilbert basis $\mathcal{H}(C)$ of a cone C is unique [120]. The Hilbert basis of C contains its generators, which are *trivial* Hilbert basis elements, the others are called *nontrivial*. It is easy to show that in the full-dimensional case, if $\det(A) = \pm 1$, then $\operatorname{cone}(A)$ has only trivial Hilbert basis elements. More generally, if the greatest common divisor of the maximum size determinants of A is 1, that is, the rows of A part of a basis of $\mathbb{Z}^n$, then cone(A) has only trivial Hilbert basis elements, and C is called *unimodular*[1].

A *triangulation* of a cone C is a collection of cones whose union is C, with the property that the intersection of any two cones in the collection is a face of each. A triangulation of C is called *Hilbert* if the generators of each cone in the triangulation are Hilbert basis elements of C, and it is *unimodular* if all its cones are unimodular. A triangulation is *regular* if there exists a polyhedron such that the triangulation is its normal fan. We refer the reader to [138, Chap. 5] and [129, Chap. 8] for more details.

3.2.2 Key results on Hilbert bases

For a set A of linearly independent integer vectors, we write gcddet(A) for the gcd of the determinants of size $|A| \times |A|$ in the matrix associated with A. We set gcddet$(\varnothing) = 1$.

The following follows from the definition of Hilbert basis elements.

Lemma 3.2.1. *Hilbert basis elements of faces of a cone are Hilbert basis elements of the cone.*

The *(fundamental) parallelepiped* generated by a finite set of vectors A is:

$$\mathcal{Z}^{<}(A) = \left\{ \sum_{a \in A} \lambda_a a \colon 0 \leqslant \lambda_a < 1 \right\}.$$

When the vectors of A are primitive, then $\mathcal{Z}^{<}(A) \cup A$ contains the Hilbert basis of the cone generated by A. Indeed, every integer point in the cone is the sum of an integer point in this parallelepiped and a nonnegative integer combination of the primitive generators.

Lemma 3.2.2 (see in [121])**.** *For a cone $C = \mathrm{cone}(A)$, we have $\mathcal{H}(C) \subseteq (\mathcal{Z}^{<}(A) \cap \mathbb{Z}^n) \cup A$.*

We can compute the number of integer points in the parallelepiped.

Lemma 3.2.3 (Folklore, see [19, Lemma 9.8])**.** *Let A be a set of linearly independent vectors of $\mathbb{Z}^n$. The number of integer points in the parallelepiped $\mathcal{Z}^{<}(A)$ is equal to* gcddet(A)*.*

The following lemma will be very useful in both proofs of Proposition 3.3.6 and Theorem 3.6.3.

Lemma 3.2.4. *Let $C = \mathrm{cone}(A)$ be a cone and $S \subseteq \mathcal{Z}^{<}(A) \cap \mathbb{Z}^n$ be a set in which no element is a non-negative integral combination of other elements of S. If the set of non-negative integer combinations of S contains $\mathcal{Z}^{<}(A) \cap \mathbb{Z}^n$, then $\mathcal{H}(C) = S \cup A$.*

Proof. First, we have $\mathcal{H}(C) \subseteq S \cup A$, as if $h \in \mathcal{H}(C) \setminus S$, then, by Lemma 3.2.2, h belongs either to $\mathcal{Z}^{<}(A) \cap \mathbb{Z}^n$ or to A. Since it is a Hilbert basis element, it cannot be expressed as an non-negative integral combination of elements in S, then it must be in A. Then, the definition of S implies $\mathcal{H}(C) = S \cup A$. $\square$

3.3 A motivating example: the cone generated by the edges of odd unicyclic graphs

Let $G = (V, E)$ be a graph with $|V| = n$ and $|E| = m$. The *incidence matrix* of G is the 0,1 matrix of size $m \times n$ whose rows are the characteristic vectors in V of its edges, and is denoted by A_G. It is proved in [33] that A_G is totally equimodular for every graph G. The goal of this section is to study the case when A_G has full row rank.

1. Note that unimodular is the same as non-singular (see Chapter 2), the terms depend on the community. Also note that when a cone is unimodular, its matrix of generators A may not be unimodular.

Example 3.3.1. Let C be a cycle of size n, that is, a graph consisting of n vertices $v_1, \dots, v_n$ and the n edges $v_i v_{i+1}$, for $i = 1, \dots, n-1$, and v_1, v_n. Then its incidence matrix is of size $n \times n$ as follows, up to reordering its rows and columns.

$$A_C = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

It is a classical exercise to compute the determinant of such a matrix. By performing a Gaussian pivot on the first column followed by another one on the second column of the incidence matrix of C, we obtain in the bottom right corner a submatrix of a cycle of size $n-2$, and a matrix with determinant 1 on the top left corner. Hence, by repeating this process, we either end up with a matrix of a cycle on two vertices with two parallel edges, that has zero determinant, or with a matrix of a cycle of size 3, which has determinant ± 2, and is invertible. We thus obtain the following.

Lemma 3.3.2. *The incidence matrix of a cycle is invertible if and only if the cycle has odd size. Moreover, in that case, it has determinant* ± 2.

The lemma follows by noting that the all one vector is the only nonzero integer point in the half-open zonotopoe.

We focus on a specific case when the matrix A_G is square invertible in $\mathbb{Q}$. A graph is *odd unicyclic* if it is connected and has a single cycle which has odd size. In particular, it has n vertices and n edges. Note that removing an edge from the odd cycle of an odd unicyclic graph yields a tree. And another well-known result is as follows.

Lemma 3.3.3. *The incidence matrix of a tree has full row rank.*

In a tree $T = (V, E)$, there is a unique *bipartition* of V into two non-intersecting subsets of vertices R and B, we write their disjoint union as $R \sqcup B$, such that every edge of E has an end in R and the other in B.

Lemma 3.3.4. *For a tree $T = (V, E)$ whose bipartition is $R \sqcup B$, we have:*

$$\{x \in \mathbb{R}^n \colon A_T x = \mathbf{0}\} = \mathbb{R}(e_R - e_B).$$

Proof. Since every edge ij has an end point in R and one in B we have $e_{ij}^\top(e_R - e_B) = 0$, for every edge ij of G, that yields $A_T(e_R - e_B) = \mathbf{0}$. The matrix A_T has size $(|V| - 1) \times |V|$ and has full row rank by Lemma 3.3.3. Hence, its kernel has dimension 1, which concludes. $\square$

Proposition 3.3.5. *Let G be a graph, then A_G is invertible in $\mathbb{Q}$ if and only if every connected component of G is an odd unicyclic graph.*

Proof. It is enough to consider connected components individually since the resulting incidence matrix will be block diagonal, up to reordering rows and columns, with each block representing the incidence matrix of a connected component. We thus suppose that G is connected

Any spanning tree of G provides $n-1$ edges on n vertices, which correspond to $n-1$ linearly independent rows of A_G by Lemma 3.3.3. Then A_G is square if and only if it it has only one more edge connecting two of these n vertices. This additional edge creates a unique cycle C. If we label the vertices of the cycle by $1, \dots, |C|$ and then label the rest of the vertices encountered by a width-first exploring starting from the vertices of the cycle, the incidence matrix of size $n \times n$ is of the form:

$$\begin{bmatrix} A_C & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \star & I_{k_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \star & \star & I_{k_2} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \mathbf{0} \\ \star & \star & \cdots & \star & I_{k_s} \end{bmatrix}, \tag{3.3.1}$$

with A_C the incidence matrix of the unique cycle, and each block $\star$ has a single nonzero column which is $\mathbf{1}$, corresponding to the root of the currently explored tree. The determinant of A_G equals $\det(A_C)\cdot\prod_{i=1}^{S}\det(I_{k_i})$, and is nonzero if and only if the cycle is of odd size by Lemma 3.3.2, that is, G is an odd unicyclic graph. $\square$

Let C_G be the cone generated by the rows of A_G, for G an odd unicyclic graph. Note the cones of edges of general graphs were studied in [137, 135].

Proposition 3.3.6. *The Hilbert basis of C_G is $A_G \cup \{e_C\}$, for e_C the characteristic vector of the odd cycle C of G.*

Proof. By Lemma 3.2.2, the nontrivial Hilbert basis elements of C_G are lattice points of the parallelepiped $\mathcal{Z}^{<}(C_G)$. By Theorem 3.2.3, since $|\det(A_G)| = 2$, there are two integer points in $\mathcal{Z}^{<}(C_G)$: one is $\mathbf{0}$ and the other one is $e_C = \frac{1}{2}\sum_{ij\in C} e_{ij}$. Since the generators form a basis of $\mathbb{R}^n$, and e_c is uniquely expressed as a non-integral sum of them, it is a Hilbert basis element. $\square$

Let us find the generators of $C_G^{\vee}$. Since C_G is simplicial, it is generated by the rows of $(A_G^{-1})^{\top}$. In particular, one way for finding the columns of A_G^{-1}, is to forget a row a of A_G and find a primitive vector that is orthogonal to all the remaining rows, while having a positive scalar product with a. Forgetting such a row corresponds to looking at the graph obtained from G after removing an edge ij. There are two cases: either ij is in the odd cycle of G and hence the resulting graph is a tree, or ij is not in the cycle and the resulting graph has two connected components: the component that contains the odd cycle is an odd unicyclic graph, and the other one is a tree. Let $G \setminus ij = (V, E \setminus \{ij\})$ be the graph obtained after removing the edge ij from G. Moreover, assume that the vertex i is contained in the unique tree of $G \setminus ij$.

Lemma 3.3.7. *For an odd unicyclic graph $G = (V, E)$, we have:*

$$C_G^{\vee} = \operatorname{cone}(\{e_R - e_B \colon R \sqcup B \textit{ is the bipartition of the unique tree of } G \setminus ij \textit{ with } i \in R\}_{ij\in E}).$$

Proof. Let C be the odd cycle of G and ij an edge of G. Let x be a generator of $C_G^{\vee}$. The graph $G \setminus ij$ either is a tree T or has two connected components consisting of an odd unicyclic graph G' and a tree T. In both cases, we can write its incidence matrix by block as follows

$$A_{G\setminus ij} = \begin{bmatrix} A_{G'} & \mathbf{0} \\ \mathbf{0} & A_T \end{bmatrix},$$

of size $(|V| - 1) \times |V|$, where G' might be empty. If G' is not empty, the condition $A_{G\setminus ij}x = 0$ implies that x is zero on the set of vertices of G', since $A_{G'}$ is invertible by Proposition 3.3.5. By Lemma 3.3.4, the condition $A_T x = 0$ implies that $x \in \mathbb{R}(e_R - e_B)$, for the bipartition $R \sqcup B$ of T where $i \in R$. Finally, since $i \in R$, the inequality $e_{ij}^{\top}x > 0$ yields the primitive generator $x = e_R - e_B$. $\square$

We now wish to obtain the Hilbert basis of $C_G^\vee$ in order to end this study. However, a quick computation shows that the determinant of the matrix of the generators of $C_G^\vee$ is 2^{k-1}, where k is the size of the odd cycle of G. This makes the use of lemma 3.2.3 more difficult since we cannot directly identify the integer points of the parallelepiped generated by the generators of $C^\vee$. This is actually a special case of the study of the Hilbert basis of te-cones done in Section 3.6, with the proofs in Section 3.8. Hence, we now delve into the general study of totally equimodular matrices with full row rank and their associated cones.

3.4 Definitions and notation

Throughout the rest of this chapter, all the entries will be rational. Moreover, we will identify a set of vectors A with the matrix, also denoted by A, whose rows are those vectors. All the considered sets will be linearly independent, and equivalently the matrices will have full row rank. A linearly independent set $A \subseteq \mathbb{R}^n$ is *full-dimensional* when it spans $\mathbb{R}^n$, or equivalently when its associated matrix is square and invertible. A subset X of A is *proper* when $\varnothing \subsetneq X \subsetneq A$. The *disjoint union* of two disjoint sets A and B is denoted by $A \sqcup B$.

We will use the following notations. Let A be an $m \times n$ matrix. The i-th row of A is denoted by A_i, its j-th column by A^j, and its coefficient (i,j) by A_i^j. The vectors $\mathbf{0}$ and $\mathbf{1}$ are the vectors with all entries being 0 and 1 of appropriate size, respectively. The matrix $\mathbf{J}$ is the all-ones matrix. The *support* $\operatorname{supp}(x)$ of a vector x is the set of its nonzero coordinates. A *pivot* will be a position $p = (i,j)$ at row index i and column index j of A whose coefficient A_i^j is not zero. Then, *p-pivoting A*, or *pivoting A with respect to p*, means dividing A_i by A_i^j, and then adding an appropriate scalar multiplication of this new row to the other rows of A, so that afterwards all the coefficients in A^j are zero, except that at row i which is now 1 [2]. The resulting matrix is called the *p-pivot* of A and will be denoted by A/p.

The matrix *trimmed* from A with respect to $p = (i,j)$ is the matrix obtained from A/p by deleting row i and column j and will be denoted by $A/\!\!/p$, the *p-trim* of A.

Then, *rescaling* a totally equimodular matrix A means dividing each row A_i by $\operatorname{eqdet}(A_i)$, and yields a 0,±1 matrix. Multiplying some rows or columns by -1 is called *resigning*. Notice that resigning preserves total unimodularity and total equimodularity.

3.5 Decomposition of totally equimodular matrices

3.5.1 Parallels between total equimodularity and total unimodularity

In the introduction, we mentioned a parallel between totally equimodular and totally unimodular matrices in terms of box-TDIness, which is how totally equimodular matrices appeared. Another connexion exists in terms of undirected graphs: the edge-vertex incidence matrix of a graph is always totally equimodular [33], and it is totally unimodular if and only if the graph is bipartite [90].

We report two new parallels between these two classes: the first one concerns pivots and trims, and the second one involves the transpose and the inverse.

Since values that appear when pivoting and trimming are values of subdeterminants of the original matrix, totally unimodular matrices can be characterized as follows.

Theorem 3.5.1 (Folklore)**.** *A matrix is totally unimodular if and only if any sequence of pivots and trims yields a* 0,±1 *matrix.*

2. Note that this is the definition of the classical Gauss-pivot, which differs from the pivot used by Seymour [124] in the decomposition theorem of totally unimodular matrices.

It turns out that this extends to totally equimodular matrices as follows. A matrix is *essentially* $0,\pm 1$ if, in each of its rows, the nonzero coefficients all have the same absolute value. In other words, multiplying each row by an appropriate coefficient yields a $0,\pm 1$ matrix. Note that totally equimodular matrices are essentially $0,\pm 1$.

Theorem 3.5.2 (Theorem 3.7.3)**.** [3] *A matrix is totally equimodular if and only if any sequence of pivots and trims yields an essentially* $0,\pm 1$ *matrix.*

Sketch of the proof. It is immediate that pivoting and trimming preserve equimodularity. This can be turned into an equivalence as follows, by considering directly the involved determinants.

Lemma 3.5.3 (Lemma 3.7.2)**.** *Let* A_i *be a* $0,\pm 1$ *row of a full row rank matrix* A*. Then,* A *is equimodular if and only if* $A /\!\!/ (i,j)$ *is equimodular for all* $j \in \operatorname{supp}(A_i)$*.*

Then, the theorem follows. □

From this, we obtain the following result on the equideterminant of $0,\pm 1$ totally equimodular matrices of full row rank.

Proposition 3.5.4. *Let* A *be a* $0,\pm 1$ *totally equimodular matrix of full row rank. Then, the equideterminant of* A *is a power of* 2*.*

Proof. This follows from a direct induction on the number of rows. Trimming preserves total equimodularity by Theorem 3.5.2. The entries of the trim of a matrix correspond to determinants of size 2×2 of the original matrix. Hence, the only possible entries of the trim are $0, \pm 1$, and ± 2, since the matrix is $0,\pm 1$. The rows which are not $0,\pm 1$ after trimming are hence $0,\pm 2$ since each of them is equimodular. We finally use the multilinearity of the determinant and factorize the ± 2's for every such rows. □

We mention that Lemma 3.5.3 will be essential throughout the proof of the decomposition theorem of the next section.

Note that Theorems 3.5.1 and 3.5.2 can be rephrased as follows: A matrix is totally unimodular (resp. totally equimodular) if and only if any sequence of pivots and trims yields a totally unimodular (resp. totally equimodular) matrix.

It is known that taking the transpose or the inverse of a matrix preserves total unimodularity [121, Page 280]. The transpose operation does not preserve total equimodularity in general, as it is highlighted for incidence matrices of graphs in [33]. Neither does taking the inverse. However, the combination of the two operations preserves total equimodularity.

Theorem 3.5.5. *Let* A *be an invertible square matrix, then* A *is totally equimodular if and only if* $(A^{-1})^\top$ *is totally equimodular.*

Proof. We will use the following [34, Theorem 2]: a cone is box-TDI if and only if the affine hull of each of its faces is described by an equimodular matrix. A full row rank matrix B is *face-defining* for a cone C when there exists a face F of C such that $\operatorname{aff}(F) = \{x : Bx = 0\}$. Note that a face has an equimodular face-defining matrix if and only if all its face-defining matrices are equimodular.

Let us prove that an invertible matrix A is totally equimodular if and only if $(A^{-1})^\top$ is totally equimodular.

Since A is invertible, every subset of rows of A forms a face-defining matrix of $C = \{x : Ax \geqslant 0\}$. Therefore, by [34, Theorem 2], A is totally equimodular if and only if C is box-TDI.

3. Here and in the next section, such a reference points to the statement and its complete proof.

By [34, Lemma 6], the latter holds if and only if the polar $C^\star = \{x : x^\top z \leqslant 0, \text{ for all } z \in C\}$ of C is box-TDI. Since C is full-dimensional and simplicial, so is its polar, which is decribed by $C^\star = \{x : (A^{-1})^\top x \geqslant 0\}$. Then, this polar is box-TDI if and only if $(A^{-1})^\top$ is totally equimodular, which concludes. □

3.5.2 The decomposition theorem

Recall that we identify a set of vectors as the matrix whose rows consist of those vectors. A linearly independent set of 0,±1 vectors is called a:

— *totally equimodular set* (*te-set*) if its associated matrix is totally equimodular,

— *totally unimodular set* (*tu-set*) if its associated matrix is totally unimodular,

— *te-lace* if it is a te-set, not a tu-set, and all its proper subsets are tu-sets,

— *te-interlace* if it is a te-set, not a tu-set, and each pair of vectors is a te-lace.

A quick check on the possible 2×2 determinant of 0,±1 matrices shows that the vectors in a te-lace of size 2 share the same support. In particular, the vectors in a te-interlace A share the same support. Therefore, up to permuting columns, we can write a te-interlace as $A = \begin{bmatrix} A' & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}$, where A' is a ±1 matrix. When appropriate, the $\mathbf{0}$ columns can be omitted. A te-interlace of size r is *thin* if its equideterminant is 2^{r-1} and *thick* if it is 2^r. A *te-brick* is either a tu-set, a te-lace, a thin te-interlace, or a thick te-interlace. Disjoint te-bricks are *mutually totally unimodular* (*mutually-tu*) when, if a set intersects several of them, contains none of the te-laces, and at most one vector of each te-interlace, then it is a tu-set.

These te-bricks are the basic structures onto which te-sets are built upon, as shown below. The property of being mutually-tu implies that the decomposition is unique.

Theorem 3.5.6 (Theorem 3.7.29)**.** *A linearly independent set of* 0,±1 *vectors A is a te-set if and only if it is the disjoint union of mutually-tu te-bricks. More precisely:*

$$A = \underbrace{U}_{\textit{tu-set}} \sqcup \underbrace{L_1 \sqcup \cdots \sqcup L_k}_{\textit{te-laces}} \sqcup \underbrace{S_1 \sqcup \cdots \sqcup S_s}_{\textit{thin te-interlaces}} \sqcup \underbrace{T_1 \sqcup \cdots \sqcup T_t}_{\textit{thick te-interlaces}}.$$

Sketch of the proof. There are two directions to be proven. First, that a te-set A is the disjoint union of mutually-tu te-bricks. The starting point is that a te-set which is not a tu-set contains a te-lace. Let $\mathcal{L}$ be the family of te-laces of A. Then, $U = A \setminus (\bigcup_{L \in \mathcal{L}} L)$ is a tu-set. Moreover, by the following key lemma, pairwise intersecting members of $\mathcal{L}$ form a te-interlace.

Lemma 3.5.7 (Lemma 3.7.26)**.** *In a te-set, if two distinct te-laces intersect, then they are both of size two and their symmetric difference is a te-lace.*

Sketch of the proof. The proof starts with a minimal counterexample, and studies the impact of various trims. First, we prove that the intersection of the two sets is a singleton. Following this, we show that one of them has size two and the other has size three. Afterwards, we study the different possibilities concerning the supports of the four involved vector, as two independent vectors form a te-lace if and only if they have the same support, and a tu-set if and only if they either coincide or are opposite on their common support. This ends up contradicting their linear independence when one of them has size three. □

Regroup the intersecting members of $\mathcal{L}$ into a family $\mathcal{I}$ of maximal te-interlaces: the remaining te-laces $\mathcal{L}'$ of $\mathcal{L}$ are pairwise disjoint and intersect no te-interlace of $\mathcal{I}$. The last ingredient to finish the decomposition is to prove that these te-interlaces are only of two types: the thin ones $\mathcal{S}$ and the thick ones $\mathcal{T}$. This is done by systematically studying the trims of te-interlaces.

By construction, no te-lace intersects distinct sets among, U, $L \in \mathcal{L}'$, $S \in \mathcal{S}$, and $T \in \mathcal{T}$, thus these sets are mutually-tu, and A is the disjoint union of mutually-tu te-bricks.

Now, there remains to prove that disjoint unions of mutually-tu te-bricks form te-sets. The proof is by induction: let A be such a set, and assume that all smaller sets which are the mutually-tu disjoint union of te-bricks are te-sets. All that remains to prove is that A is equimodular.

We start by proving that every $A' \subsetneq A$ also is a mutually-tu disjoint union of te-bricks. Then, we study the impact of trimming on the different te-bricks involved. Let B be a te-brick of A and $a = A_i$ a row of A. We first prove the following thanks to the mutually-tu property: if $a \notin B$, then the set obtained from B after (i,j)-trimming A, for any column j of A with $A_i^j \neq 0$, is of the same type as B. If $a \in B$, there are several cases: if B is a tu-set or a te-lace of size two, then, after rescaling, $B /\!/ (i,j)$ is a tu-set; if B is a te-lace of size at least three, then $B /\!/ (i,j)$ is a te-lace.

Now, if A contains a row which is in no te-interlace, this row is used to trim A, and to retrieve its equimodularity thanks to the above facts and Lemma 3.5.3. Otherwise, each row of A is in a te-interlace, and we need an additional property: trimming and then rescaling a te-interlace yields a te-set in which there are no te-interlaces. By the first direction of the theorem, it becomes the disjoint union of a tu-set and te-laces which are mutually-tu. We then prove that the set resulting from A by rescaling such a trim is the mutually-tu disjoint union of te-bricks, and hence is a te-set by the induction hypothesis. In particular, fixing a row and performing all possible trims yields an equimodular matrix, and hence A is equimodular, again by Lemma 3.5.3. $\square$

Theorem 3.5.6 raises a complexity question: *Can the decomposition be obtained in polynomial time?* Finding a candidate for the decomposition into disjoint te-bricks can be done in polynomial time, because testing total unimodularity can be done in polynomial time [124]. However, deciding if these te-bricks are mutually-tu seems challenging.

Once the decomposition is known, so is the equideterminant.

Corollary 3.5.8. *If A is a te-set that decomposes as in Theorem 3.5.6, we have* $\mathrm{eqdet}(A) = 2^{k+\sum_i(|S_i|-1)+\sum_j |T_j|}$.

Together with [1, Theorem 1.7], this yields the following [4].

Corollary 3.5.9. *For a totally equimodular matrix $A \in \{0,\pm1\}^{m\times n}$ and $b \in \mathbb{Z}^m$, the system $Ax \leqslant b$ is totally dual dyadic.*

We mention that the full row rank hypothesis is essential to derive the above decomposition theorem, as the key lemma fails without it. The situation might be dramatically intricated as shows the example in Figure 3.1. This raises the question: *Is there a decomposition theorem for general totally equimodular matrices?*

3.5.3 A conjecture and connections with other classes of matrices

Supported by the fact that a brute force enumeration by computer showed that there are no thick te-interlaces of size 8, we conjecture the following.

Conjecture 3.5.10. There are only two full-dimensional thick te-interlaces, up to resigning or permuting rows and columns:

4. Note that the full row rank assumption is dropped here.

$$M = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Figure 3.1 – A totally equimodular matrix M without full row rank, in which the te-laces are $\{1, 2, 3, 4\}$, $\{3, 4, 5, 6\}$, $\{1, 3, 5\}$, and $\{2, 3, 6\}$, and pairwise intersect. Note that there are even intersections of size two.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \text{and} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 \end{bmatrix}.$$

This conjecture connects with other classes of matrices, beyond the parallel between totally equimodular and totally unimodular matrices.

First, it turns out that there are further relations between each types of te-bricks. Let A be a square invertible matrix. Then, A is a thin te-interlace if and only if $\left(\frac{1}{2}A\right)^{-1}$ is a te-lace. In other words, thin te-interlaces are essentially the inverses of minimally non-totally unimodular matrices, where a matrix is *minimally non-totally unimodular* if it is not totally unimodular, but all its proper submatrices are totally unimodular. Moreover, A is a thick te-interlace if and only if $\left(\frac{1}{4}A\right)^{-1}$ is a thick te-interlace.

More generally, te-interlaces encode classes of 0,1 matrices studied by Truemper in [132] and [133]. Let A be a 0,1 matrix and i a row index of A. The *row-i complement* of A is the 0,1 matrix obtained from A whose i-th row is unchanged and whose i'-th rows are $A_i + A_{i'} \pmod 2$, for $i' \neq i$. *Column-j complements* are defined similarly.

Example 3.5.11. The identity matrix of size 4, the row-1 complement, followed by the column-1 complement.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

The *complement orbit* of A is the set of 0,1 matrices obtained from A by any sequence of complement operations. A 0,1 matrix A is *complement totally unimodular* if its complement orbit contains only totally unimodular matrices, and *complement minimally non-totally unimodular* if its complement orbit contains only minimally non-totally unimodular matrices. Let A be a ± 1 matrix that we write, up to resigning or permuting rows and columns, as

$$A = \begin{bmatrix} 1 & \mathbf{1}^\top \\ \mathbf{1} & \mathbf{J} - 2B \end{bmatrix}, \tag{3.5.1}$$

for some 0,1 matrix B. Then, up to resigning and rescaling, the trims of A run across the complement orbit of B. An ingredient of the proof of Theorem 3.5.6 is that, in (3.5.1), A is a thin te-interlace if and only if B is complement totally unimodular, and A is a thick te-interlace if and only if B is complement minimally non-totally unimodular.

3.6 Hilbert triangulation of te-cones

3.6.1 Hilbert basis of te-cones

In this section, we explicitly identify the Hilbert basis of te-cones. Recall that te-cones are simplicial cones generated by te-sets. By Theorem 3.5.6, a te-cone is the Minkowski sum of cones generated by mutually-tu te-bricks whose union is linearly independent. Thanks to the following two lemmas, we can focus on each te-brick separately.

Lemma 3.6.1. *Let A and B be two mutually-tu te-bricks whose union is linearly independent. Then we have $|\mathcal{Z}^{<}(A \cup B)| = |\mathcal{Z}^{<}(A)| \cdot |\mathcal{Z}^{<}(B)|$. That is, the number of integer point within the parallelepiped generated by the union equals the product of the number of integer points within the parallelepipeds generated by each set.*

Proof. we have $\operatorname{eqdet}(A \cup B) = \operatorname{eqdet}(A) \operatorname{eqdet}(B)$, so we can use Lemma 3.8.2. □

A direct consequence is.

Lemma 3.6.2 (Lemma 3.8.3)**.** *The Hilbert basis of the Minkowski sum of cones generated by mutually totally unimodular te-bricks whose union is linearly independent is the union of the Hilbert basis of each.*

Recall that we identify a set of vectors $A = \{a^1, \dots, a^m\}$ with the matrix whose rows are the a^i's. Then, here is the Hilbert basis of each type of te-brick.

Theorem 3.6.3 (Theorem 3.8.6)**.** *Let $A = \{a^1, \dots, a^m\}$ be a te-brick and $C = \operatorname{cone}(A)$.*

1. *If A is a tu-set, then $\mathcal{H}(C) = A$.*
2. *If A is a te-lace, then $\mathcal{H}(C) = A \cup \{\frac{1}{2} \sum_j a^j\}$.*
3. *If A is a thin te-interlace, then $\mathcal{H}(C) = A \cup \left\{\frac{1}{2}(a^i + a^j)\right\}_{1 \leqslant i < j \leqslant m}$.*
4. *If A is a thick te-interlace, then one of the following holds:*
   a. $\mathcal{H}(C) = A \cup \left\{\frac{1}{2}(a^i + a^j)\right\}_{1 \leqslant i < j \leqslant m} \cup \left\{\frac{1}{4} \sum_j a^j\right\}$,
   b. $\mathcal{H}(C) = A \cup \left\{\frac{1}{2}(a^i + a^j)\right\}_{1 \leqslant i < j \leqslant m} \cup \left\{\frac{3}{4}a^i + \frac{1}{4} \sum_{j \neq i} a^j\right\}_{i \in \{1, \dots, m\}}$.

   Moreover, the number of 1's and -1's in each column of A have the same parity $p \in \{0, 1\}$, and 4.a occurs if and only if $m \equiv 2p \pmod 4$.

Sketch of the proof.

1. Simplicial cones generated by tu-sets have no nontrivial Hilbert basis elements.
2. The only nontrivial Hilbert basis element of a simplicial cone generated by a te-lace is the half sum of its generators.

3-4. For cones generated by te-interlaces, we first recall that the nontrivial Hilbert basis elements of a cone lies within the parallelepiped generated by the generators of the cone. We know that this number of points within this parallelepiped is the equideterminant of A. Let S be the set of vectors in Case 3., 4.a, or 4.b, from which we remove A. In each case, we find out that the number of distinct non-negative integer combinations of the vectors in S within the parallelepiped is also equal to the equideterminant of A. This implies that $\mathcal{H}(C) \subseteq S \cup A$. Finally, $S = \mathcal{H}(C)$ since none of the vectors of S is a non-negative integer combination of the others.

□

Remark 3.6.4 (See Section 3.8.1 for more details)**.** For Case *3.*, the integer points in the parallelepiped are the 1/2 sums of an even numbers of a^i's. In Case *4.*, we have in addition in Case *a.* for any partition into two even sets $\{A_1, A_2\}$, 3/4 times the sum on A_1 to which we add 1/4 the sum on A_2, which is indeed, an integer, and their number is also 2^{m-1}, so we have a parallelepiped of size 2^m, as it should. And in Case *b.* we have the same for odd partitions $\{A_1, A_2\}$.

3.6.2 Regular unimodular Hilbert triangulation of te-cones

In this section, we prove that cones generated by te-sets with no thick te-interlace of size greater than six admit a regular unimodular Hilbert triangulation. Thanks to Theorems 3.5.6 and 3.6.3, and to the lemma below, it is enough to provide a regular unimodular Hilbert triangulation for each type of te-brick involved.

The *join* of two triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ of two cones generated by disjoint sets whose union is linearly independent is the triangulation $\mathcal{T}_1 * \mathcal{T}_2 = \{C_1 + C_2 \colon C_1 \in \mathcal{T}_1, C_2 \in \mathcal{T}_2\}$.

Lemma 3.6.5 (Lemma 3.8.12)**.** *The join of the regular unimodular Hilbert triangulations of cones generated by disjoint te-bricks whose union is linearly independent is a regular unimodular Hilbert triangulation of their Minkowski sum.*

Sketch of the proof. By Lemma 3.6.2, this construction gives a Hilbert triangulation. The te-bricks are disjoint and their union is linearly independent, so this join triangulation is regular by [87, Sect. 2.3.2]. Moreover, each cone in the join is unimodular by the fact the te-bricks are mutually totally unimodular. □

If Conjecture 3.5.10 is true, then the following theorem together with Lemma 3.6.5 implies that all te-cones admit a regular unimodular Hilbert triangulation.

Theorem 3.6.6 (Theorem 3.8.13)**.** *Let A be a te-set without thick te-interlace of size greater than six. Then,* $\mathrm{cone}(A)$ *has a regular unimodular Hilbert triangulation.*

Sketch of the proof. By Theorem 3.5.6, A is the disjoint union of mutually-tu te-bricks, and then $C = \mathrm{cone}(A)$ is the Minkowski sum of the cones generated by these te-bricks. By Lemma 3.6.5, there remains to triangulate the cones generated by each of the four te-bricks. We thus suppose that $A = \{a^1, \dots, a^m\}$ is a te-brick. We organize the proof according to the different cases for the Hilbert basis of Theorem 3.6.3.

1. When A is a tu-set, the regular unimodular Hilbert triangulation is the cone.
2. When A is a te-lace, the stellar triangulation at $h = \frac{1}{2}\sum_j a^j$, namely the one formed by the m cones generated by h and $m-1$ generators among m, is regular since its coincides with the strong pulling at h which preserved regularity [87, Lemma 2.1]. Moreover, a determinant computation yields the unimodularity. Finally all the cones are generated by Hilbert basis elements.
3. Suppose A is a thin te-interlace. Inspired by the regular triangulation in [61], we start this case with some definitions. A *spanning* subgraph of a graph $G = (V, E(G))$ is a connected graph $H = (V, F)$ with $F \subseteq E(G)$. Let $\mathring{K}_m$ be a complete graph with m vertices to which we added an edge ii called a *loop* at each vertex i. The edges of $\mathring{K}_n$ encode the Hilbert basis elements of C as follows: an edge ij represents $\frac{1}{2}(a^i + a^j)$. The latter is a^i for a loop ii. Embed $\mathring{K}_m$ as a convex m-gon in $\mathbb{R}^2$, with clockwise labeled vertices $v_1, \dots, v_m$, edges ij embedded as line segments $[v_i, v_j]$, for each $i \neq j$, and loops ii as circles outside the m-gon, intersecting the m-gon only at v_i. We say that two distinct edges *meet* if the associated segments of the convex m-gon labeled by these numbers intersect. This happens

either if they have a common end, or if the edges are ik and jl with $i < j < k < l$. A *stellar cycle* of this embedding $\mathring{K}_m$ is a spanning subgraph with m pairwise intersecting edges or loops. Let $\mathcal{S}_m$ denote the set of stellar cycles of $\mathring{K}_m$.

Claim 3.6.7. *The cones* $C_S = \operatorname{cone}(\frac{1}{2}(a^i + a^j) \colon ij \in E(S))$, *for all* $S \in \mathcal{S}_m$, *form a regular unimodular Hilbert triangulation of* C.

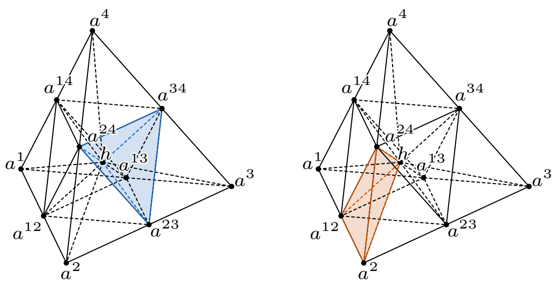
Sketch of the proof. These cones are all generated by Hilbert basis elements of C. Up to a linear transformation by $(2A^{-1})^\top$, it is the regular triangulation of the second hypersimplex given in [61] to which we attach the simplex composed of $2e^i$ and the simplicial facet $\{x_i = 1\}$, for $i = 1, \dots, m$. Attaching these simplices preserves regularity and determinant computations yield unimodularity. □

4. Suppose that A is a thick te-interlace. There are four cases: $m = 4$ or 6 and Case 4.a or 4.b. We used Polymake [76] to check regularity and a simple algorithm to check unimodularity. See Figures 3.2 and 3.3 for the case $m = 4$.

   a. By taking the triangulation of Case 3. restricted to the boundary of C and adding to each of its cones the generator $h = \frac{1}{4}\sum_j a^j$, we obtain a unimodular triangulation of C, see Figure 3.2.

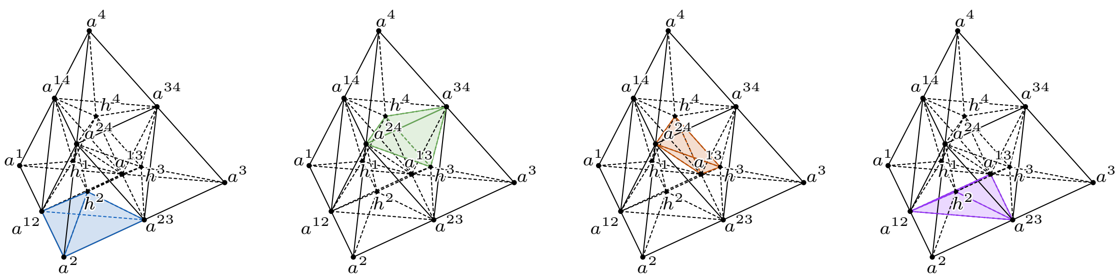
   b. Here, we set $h^i = \frac{3}{4}a^i + \frac{1}{4}\sum_{j \neq i} a^j$, for $i = 1, \dots, m$, and the triangulation has a more complex structure, see Figure 3.3.

The regular unimodular Hilbert triangulation in the case $m = 4$ are displayed in Figures 3.2 and 3.3. We define $a^{ij} = \frac{1}{2}(a^i + a^j)$, for $i \neq j$. Up to symmetry, the cones in the triangulations are generated by the vertices of the colored tetrahedra in each case.



4 of this type. 12 of this type.

Figure 3.2 – Case 4.a.



12 of this type. 8 of this type. 4 of this type. 4 of this type.

Figure 3.3 – Case 4.b.

□

In Section 3.5, we raised the question of the existence of a decomposition theorem without the full row rank hypothesis. One could hope nevertheless that Theorem 3.6.3 provides a Hilbert basis in the general case, for instance by applying it to each simplicial subcone, which would be a te-cone. This is not the case as the following example shows. For the cone $C = \mathrm{cone}(M)$ generated by the rows of the matrix M given in Figure 3.1 at the end of Section 3.5, this strategy yields four vectors, namely the half-sum of each te-lace:

$$h_1^\top = \begin{bmatrix}2 & 1 & 1 & 1\end{bmatrix}, h_2^\top = \begin{bmatrix}1 & 1 & 1 & 2\end{bmatrix}, h_3^\top = \begin{bmatrix}1 & 1 & 0 & 1\end{bmatrix}, h_4^\top = \begin{bmatrix}1 & 0 & 1 & 1\end{bmatrix}.$$

Yet, h_1 and h_2 are not Hilbert basis elements of C as $h_1 = h_4 + M_1^\top$ and $h_2 = h_3 + M_6^\top$. Independently from the existence of a decomposition theorem, the following question remains open: *In general, which integer decomposition properties do cones generated by totally equimodular matrices satisfy?*

3.7 Proofs of the results of Section 3.5: decomposition

We first provide notations and remarks that will be used throughout the next two sections.

Let A be an $m \times n$ matrix. For subsets $I \subseteq \{1,\dots,m\}$ and $J \subseteq \{1,\dots,n\}$ we write A_I the $|I| \times n$ submatrix of A whose rows are indexed by I, A^J the $m \times |J|$ submatrix of A whose columns are indexed by J, and $A_I^J = (A_I)^J = (A^J)_I$ the associated $|I| \times |J|$ submatrix of A. Whenever I or J is a singleton, we omit the curly brackets. For instance, choosing of row index i and a column index j, we write $A_i = A_{\{i\}}$ the i-th row vector of A, and $A^j = A^{\{j\}}$ the j-th column vector of A. We write $A_{\widehat{i}} = A_{\{1,\dots,i-1,i+1,\dots,m\}}$ and $A^{\widehat{j}} = A_{\{1,\dots,j-1,j+1,\dots,n\}}$. Moreover, we will write $A^{<j} = A^{\{1,\dots,j-1\}}$ and $A^{>j} = A^{\{j+1,\dots,m\}}$, with $A_{<i}$ and $A_{>i}$ defined similarly. For a transposition (ij), we define $\Pi_{(ij)}$ the matrix obtained from the identity matrix $\mathbf{I}_n$ by exchanging the coefficients (i,i) and (j,j) with the coefficients (i,j) and (j,i), respectively.

For $a \in \mathbb{Q}^n$, we define $\mathrm{diag}(a)$ the square matrix whose i-th diagonal coefficient is a_i, and whose other coefficients are zeroes. When a is ± 1, $\mathrm{diag}(a)$ is called a *signing matrix*.

To simplify notation, we will often directly write $r \in A$ to denote a row $r = A_i$ of A. Then, a pivot (r,j) with $j \in \mathrm{supp}(r)$, is the same as the pivot (i,j) defined in Section 3.4. When A is totally equimodular, we denote by $\widehat{A}$ the rescaled version of A obtained by dividing each row r of A by $\mathrm{eqdet}(r)$, which is a 0,$\pm$1 matrix.

We say that *r and s coincide on their common support* when $r_{\mathrm{supp}(r)\cap\mathrm{supp}(s)} = s_{\mathrm{supp}(r)\cap\mathrm{supp}(s)}$. They are *opposites on their common support* when r and $-s$ coincide on their common support. For two vectors r and s which coincide on they common support, $r \triangle s$ denotes the vector obtained from $r+s$ by setting the coordinates in $\mathrm{supp}(r) \cap \mathrm{supp}(r)$ to 0.

We will use intensively the following easy remarks about full row rank sets of size two, which come from the fact that $\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$ is the only 2×2 minimally non-totally unimodular matrix, up to resigning rows and columns:

- $\{r,s\}$ is a tu-set if and only if $\mathrm{supp}(r) \neq \mathrm{supp}(s)$ and r and s either coincide or are opposites on their common support.
- $\{r,s\}$ is a te-lace if and only if $\mathrm{supp}(r) = \mathrm{supp}(s)$ and $r \neq \pm s$.
- In a te-interlace, all the rows have the same support.
- If r and s coincide on their common support and $c \in \mathrm{supp}(r) \cap \mathrm{supp}(s)$, then we have $s' = \{r,s\} /\!\!/ (r,c) = s - r = s \triangle (-r)$. Hence $\mathrm{supp}(s') = \mathrm{supp}(s) \triangle \mathrm{supp}(r)$.
- Trimming a te-interlace yields a matrix of the form $2B$, where B can be made 0,1 by resigning rows and columns.

3.7.1 Pivoting and trimming in te-bricks

In this section, we provide preliminary results about the impact of pivoting and trimming on equimodularity and te-bricks.

First, the following is immediate.

Lemma 3.7.1. *Let A be a full row rank $m \times n$ matrix and (i, j) be a pivot of A. Then, A is equimodular if and only if $A/(i, j)$ is equimodular.*

Proof. The row operations involved in the (i, j)-pivot divide all the $m \times m$ determinants of A by A_i^j. $\square$

The previous result has a counterpart in terms of trims.

Lemma 3.7.2 (Lemma 3.5.3)**.** [5] *Let r be a 0,$\pm$1 row of a full row rank $m \times n$ matrix A. Then, A is equimodular if and only if $A/\!\!/(r, j)$ is equimodular for all $j \in \text{supp}(r)$.*

Proof. The "only if" part follows from Lemma 3.7.1, in fact, for $J \sqcup \{j\} \subseteq \{1, \ldots, n\}$ of size m, we have $\det\left((A/(r, j))^{J \sqcup \{j\}}\right) = \pm \det\left((A/\!\!/(r, j))^J\right)$.

Suppose now that, for all $k \in \text{supp}(r)$, $A/\!\!/(r, k)$ is equimodular, and let $d_k = \text{eqdet}(A/\!\!/(r, k))$. We have $d_k \neq 0$ for all $k \in \text{supp}(r)$.

Let $i \neq j$ be in $\text{supp}(r)$. Note that both A^i and A^j are nonzero. If $A^i = \pm A^j$, then trimming with respect to (r, i) and (r, j) yield the same determinants, up to the sign, thus $d_i = d_j$. Otherwise, there exists a submatrix D of A containing both A^i and A^j which is invertible. Let $D_i = D/\!\!/(r, i)$ and $D_j = D/\!\!/(r, j)$. By definition, $|\det(D_i)| = d_i$ and $|\det(D_j)| = d_j$. Now, since r is 0,$\pm$1, and D_i and D_j are respectively trimmed from D by pivoting with respect to (r, i) and (r, j), we have $|\det(D_i)| = |\det(D)| = |\det(D_j)|$. Thus $d_i = d_j$, and all nonzero d_k's have the same value.

Thoses values are presicely the values of the $n \times n$ nonzero determinants of A, thus A is equimodular. $\square$

Lemmas 3.7.1 and 3.7.2 imply the following, which is an equivalent formulation of Theorem 3.5.2.

Theorem 3.7.3 (Theorem 3.5.2)**.** *A matrix is totally equimodular if and only if any sequence of pivots and trims yields a totally equimodular matrix.*

We provide additional properties about trimming in te-laces and te-interlaces.

Lemma 3.7.4. *Trimming a te-lace of size at least three yields a te-lace.*

Proof. This holds because trimming a minimally non-totally unimodular matrix of size at least three yields a minimally non-totally unimodular matrix, as trimming in a 0,$\pm$1 square matrix preserves the determinant of the whole matrix, and total unimodularity of the proper submatrices. $\square$

Lemma 3.7.5. *After trimming and rescaling a te-interlace, there are no te-laces of size two. In particular, there are no te-interlaces.*

Proof. Let $p = (r, j)$ be a pivot of a te-interlace I, and let $I' = I/\!\!/p$. Recall that all the rows of I have the same support. Up to resigning rows and columns, we may assume that r has only ones on its support, and that $t_j = -1$ for all $t \in I \setminus r$. Then, I' is composed of the rows $t + r$, for all $t \in I \setminus r$, to which we deleted the j-th column. Since every such t is 0,$\pm$1, the vectors in I' have entries only in $\{0, 2\}$. Hence, rescaling I' yields a 0,1 matrix which contains no te-lace of size two, as there are no 0,1 minimally non-totally unimodular matrices of size 2×2. $\square$

5. Here and in the next section, such a reference points to the same statement in the first part.

Lemma 3.7.6. *Let $X = \{\ell, r, s\}$ be a te-set with $\{\ell, r\}$ a tu-set and $|\operatorname{supp}(z)| \geqslant 2$ for all $z \in X$. Let $\{r', s'\} = (X/(\ell, j)) \setminus \{\ell\}$ for some $j \in \operatorname{supp}(r) \cap \operatorname{supp}(\ell)$. If $\operatorname{supp}(r') = \operatorname{supp}(s')$, then either $\{r, s\}$ or $\{\ell, r, s\}$ is a te-lace.*

Proof. Suppose that $\{r, s\}$ is not a te-lace, that is, it is a tu-set and in particular $\operatorname{supp}(r) \neq \operatorname{supp}(s)$. Let us prove that then $\{\ell, r, s\}$ is a te-lace.

Since $\{\ell, r\}$ and $\{r, s\}$ are both tu-sets, up to resigning rows, we may assume that r and ℓ coincide on their common support, as well as r and s. Moreover, $r' = r\triangle(-\ell)$. Since X has full row rank, $\{r', s'\}$ has full row rank, and hence is a te-lace as $\operatorname{supp}(r') = \operatorname{supp}(s')$.

If $s' = s$, then $\operatorname{supp}(s) \neq \operatorname{supp}(\ell)$ hence $\{s, \ell\}$ is a tu-set. Since $X /\!\!/ (\ell, j)$ is a te-lace, X is not tu. But all its proper subsets are tu, thus X is a te-lace. If $s' \neq s$, then $s' = s\triangle(-\ell)$ up to resigning. But then $\operatorname{supp}(r) = \operatorname{supp}(s)$, contradicting the assumption that $\{r, s\}$ is a tu-set. □

3.7.2 Two kinds of te-interlaces

This section is devoted to the proof that only two types of te-interlaces exist: the thin and thick ones. Before, let us provide a few technical results that we shall use.

Minimally non-totally unimodular matrices

Recall that a matrix is minimally non-totally unimodular when it is not totally unimodular but all its proper submatrices are. First, Camion [30] provides the following about minimal non-totally unimodular matrices.

Theorem 3.7.7 ([30]). *Let A be a minimally non-totally unimodular matrix, then $\det(A) = \pm 2$ and A^{-1} has only $\pm\frac{1}{2}$ entries. Furthermore, each row and each column of A has an even number of nonzeroes and the sum of all entries in A equals $2 \pmod 4$.*

As a consequence we obtain the following useful lemma.

Lemma 3.7.8. *Adding a column x to a minimally non-totally unimodular matrix A yields a totally equimodular matrix A' if and only if x is $\mathbf{0}$ or $\pm A^j$ for some column A^j of A.*

Proof. The "if" part is direct. Let us prove the "only if" part. When A is of size 2, the result is immediate since there are only four ± 1 vectors in $\mathbb{R}^2$ and up to resigning, the only minimally non-totally unimodular matrix of size 2 is $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. Let A be of size greater than 2 and $x \neq \mathbf{0}$. There exists a column A^j of A such that the matrix B obtained from A' by removing column A^j is invertible. Since A' is totally equimodular and A is minimally non-totally unimodular, B is minimally non-totally unimodular. By the second part of Theorem 3.7.7, each row of both A and B has an even sum. Consequently, x and A^j have the same support. Now, if $x \neq \pm A^j$, then, up to resigning, A' contains $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ of determinant 2 as a proper submatrix, which contradicts the total equimodularity of A'. □

The following follows from the definition of total equimodularity and the fact that if a 0,±1 matrix is not totally unimodular, then it contains a minimally non-totally unimodular matrix.

Lemma 3.7.9. *A te-set which is not a tu-set contains a te-lace.*

Complement matrices

During the proof of Lemma 3.7.5, we observed that a single trim in a te-interlace yields a 0,1 matrix, up to rescaling and resigning. We introduce specific classes of 0,1 matrices that arise from trims of te-interlaces. Notably, these matrices generalize complement totally unimodular and complement minimally non-totally unimodular matrices, which are thoroughly studied in Truemper's book [133] and in [132].

In $\{0,1\}$, we define the following operation which corresponds the mod 2 sum: $a \oplus b = a + b$ (mod 2). More explicitly, we have $0 \oplus 0 = 1 \oplus 1 = 0$ and $1 \oplus 0 = 0 \oplus 1 = 1$. Let A be a 0,1 matrix of size $m \times n$, and i a row index of A. The *row-i complement* $A_{[i]}$ of A is the 0,1 matrix whose i'-th row is

$$\begin{cases} A_i, & \text{if } i' = i, \\ A_{i'} \oplus A_i, & \text{otherwise.} \end{cases}$$

Column-j complements $A^{[j]}$ are defined similarly. Some interesting properties arise from row or column complement operations. These operations are involutions: $(A_{[i]})_{[i]} = A = (A^{[j]})^{[j]}$. Any row and column complement operation commute: $(A_{[i]})^{[j]} = (A^{[j]})_{[i]}$. Moreover, up to permutation of rows, respectively of columns, we have $(A_{[i]})_{[i']} = \Pi_{(ii')} A_{[i']}$, respectively $(A^{[j]})^{[j']} = A^{[j']} \Pi_{(jj')}$. For convenience, we will write $A^{[0]}_{[0]} = A$, $A^{[0]}_{[i]} = A_{[i]}$, $A^{[j]}_{[0]} = A^{[j]}$, and $A^{[j]}_{[i]} = (A_{[i]})^{[j]} = (A^{[j]})_{[i]}$, for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. Hence, up to permutation of rows and columns, there are $(m+1) \cdot (n+1)$ matrices obtainable from A using row and column complement operations, and the set composed of these matrices is called the *complement orbit* $\mathcal{O}(A)$ of A:

$$\mathcal{O}(A) = \mathcal{O}(A^{[j]}_{[i]}) = \left\{ A^{[j]}_{[i]} : i \in \{0, \dots, m\}, j \in \{0, \dots, n\} \right\}.$$

A 0,1 matrix is *complement totally equimodular* if all the matrices in its complement orbit are totally equimodular. Special cases are *complement totally unimodular* matrices, for whom the complement orbit consists of totally unimodular matrices only, and *complement minimally non-totally unimodular* whose complement orbit is composed of minimally non-totally unimodular matrices.

Truemper [132] proved the following, for which we provide a shorter proof.

Theorem 3.7.10 ([132]). *Complement minimally non-totally unimodular matrices are of odd size.*

Proof. Let A be a 0,1 minimally non-totally unimodular matrix of even size n, and $A^j_i \neq 0$. By Theorem 3.7.7, the column A^j of A has even support and has an even number of zeroes, since n is even. Performing the row-i complement operation replaces the support of A^j by its complement, except at coefficient i. This yields a column of odd support, hence $A_{[i]}$ is not minimally non-totally unimodular by Theorem 3.7.7, and A is not complement minimally non-totally unimodular. □

Let A be a ± 1 matrix of size $m \times n$. We define $\operatorname{neg}(A)$ to be the 0,1 matrix whose support is the location of the -1 entries of A. In particular $A = \mathbf{J} - 2\operatorname{neg}(A)$. Note that we have $\operatorname{neg}(A_i) = \operatorname{neg}(A)_i$ and $\operatorname{neg}(A^j) = \operatorname{neg}(A)^j$, for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. For B a 0,1 matrix, we denote by $\overline{B} = \mathbf{J} - B$ the 0,1 matrix whose support is the complement of that of B. Since A is a ± 1 matrix, we have $\operatorname{neg}(-A) = \overline{\operatorname{neg}(A)}$.

Lemma 3.7.11. *Let A be a ± 1 matrix of size $m \times n$. Then, for every $\varepsilon \in \{\pm 1\}^m$, we have:*

$$\operatorname{neg}(A \operatorname{diag}(\varepsilon)) = \begin{bmatrix} \operatorname{neg}(A)_1 \oplus \operatorname{neg}(\varepsilon)^\top \\ \vdots \\ \operatorname{neg}(A)_m \oplus \operatorname{neg}(\varepsilon)^\top \end{bmatrix},$$

and for every $\mu \in \{\pm 1\}^n$, we have:

$$\operatorname{neg}(\operatorname{diag}(\mu) A) = \begin{bmatrix} \operatorname{neg}(A)^1 \oplus \operatorname{neg}(\mu) & \cdots & \operatorname{neg}(A)^n \oplus \operatorname{neg}(\mu) \end{bmatrix}.$$

Proof. This comes from $\operatorname{neg}(x \cdot y) = \operatorname{neg}(x) \oplus \operatorname{neg}(y)$, for all $x, y \in \{\pm 1\}$. $\square$

Up to resigining rows and columns, let us write

$$A = \begin{bmatrix} 1 & \mathbf{1}^\top \\ \mathbf{1} & \mathbf{J} - 2B \end{bmatrix}, \tag{3.7.1}$$

for a 0,1 matrix B called a *core* of A. Note that we have $\operatorname{neg}(A) = \begin{bmatrix} 0 & \mathbf{0}^\top \\ \mathbf{0} & B \end{bmatrix}$. Finally, we write $A \simeq A'$ when A equals A' up to permutation of rows and columns.

In the following, we relate trims of ± 1 matrices with the complement orbit of their core.

Lemma 3.7.12. *Let A be a ± 1 matrix of size $(m+1) \times (n+1)$, and B a core of A as in* (3.7.1)*. Then, for any pivot $p = (i+1, j+1)$ of A, with $0 \leqslant i \leqslant m$ and $0 \leqslant j \leqslant n$, we have*

$$A /\!\!/ p \simeq -2 A_{i+1}^{j+1} \operatorname{diag}(\varepsilon) B_{[i]}^{[j]} \operatorname{diag}(\mu), \tag{3.7.2}$$

where $\operatorname{diag}(\varepsilon)$ and $\operatorname{diag}(\mu)$ are signing matrices with $\varepsilon = \mathbf{1} - 2B^j$ and $\mu = \mathbf{1}^\top - 2B_i$, and the convention $B^0 = \mathbf{0}$ and $B_0 = \mathbf{0}^\top$.

Proof. First, note that $A /\!\!/ (1,1) = -2B = -2A_1^1 \operatorname{diag}(\mathbf{1} - 2B^0) B_{[0]}^{[0]} \operatorname{diag}(\mathbf{1}^\top - 2B_0)$. Similarly, with a matrix of the form

$$A' = \begin{array}{c c} & \begin{array}{ccc} \phantom{\mathbf{J}-2D} & j & \phantom{\mathbf{J}-2E} \end{array} \\ i & \begin{bmatrix} \mathbf{J} - 2D & \mathbf{1} & \mathbf{J} - 2E \\ \mathbf{1}^\top & 1 & \mathbf{1}^\top \\ \mathbf{J} - 2F & \mathbf{1} & \mathbf{J} - 2G \end{bmatrix} \end{array}, \text{ we have } \operatorname{neg}(A') = \begin{array}{c c} & \begin{array}{ccc} & j & \end{array} \\ i & \begin{bmatrix} D & \mathbf{0} & E \\ \mathbf{0}^\top & 0 & \mathbf{0}^\top \\ F & \mathbf{0} & G \end{bmatrix} \end{array}, \tag{3.7.3}$$

and we get $A' /\!\!/ (i,j) = -2 \begin{bmatrix} D & E \\ F & G \end{bmatrix}$.

Claim 3.7.13. *For $A \in \{\pm 1\}^{m \times n}$, $\varepsilon \in \{\pm 1\}^m$, $\mu \in \{\pm 1\}^n$, $i \in \{1, \ldots, m\}$, and $j \in \{1, \ldots, n\}$, we have:*

$$(\operatorname{diag}(\varepsilon) A \operatorname{diag}(\mu)) /\!\!/ (i,j) = \operatorname{diag}(\varepsilon_{\widehat{i}}) (A /\!\!/ (i,j)) \operatorname{diag}(\mu_{\widehat{j}}). \tag{3.7.4}$$

Proof. Since A is ± 1, we have $A_i^j = \frac{1}{A_i^j}$, and the classical formula for pivoting yields $A /\!\!/ (i,j) = A_{\widehat{i}}^{\widehat{j}} - A_{\widehat{i}}^j A_i^j A_i^{\widehat{j}}$. By applying this formula to $M = \operatorname{diag}(\varepsilon) A \operatorname{diag}(\mu)$, and since ε and μ are ± 1, we have:

$$\begin{aligned}M/\!\!/(i,j) &= M_{\widehat{i}}^{\widehat{j}} - M_i^j M_{\widehat{i}}^j M_i^{\widehat{j}}\\ &= \operatorname{diag}(\varepsilon_{\widehat{i}})(A_{\widehat{i}}^{\widehat{j}})\operatorname{diag}(\mu_{\widehat{j}}) - \varepsilon_i A_i^j \mu_j(\operatorname{diag}(\varepsilon_{\widehat{i}})A_{\widehat{i}}^j\mu_j)(\varepsilon_i A_i^{\widehat{j}}\operatorname{diag}(\mu_{\widehat{j}}))\\ &= \operatorname{diag}(\varepsilon_{\widehat{i}})\left(A_{\widehat{i}}^{\widehat{j}} - A_i^j A_{\widehat{i}}^j A_i^{\widehat{j}}\right)\operatorname{diag}(\mu_{\widehat{j}})\\ &= \operatorname{diag}(\varepsilon_{\widehat{i}})(A/\!\!/(i,j))\operatorname{diag}(\mu_{\widehat{j}}).\end{aligned}$$

Thus, the claim is proved. □

Our strategy is to sign A in order to obtain a matrix A' of the form (3.7.3), and then perform the (i,j)-trim, which is the matrix $\operatorname{neg}(A')$ after removing the i-th row and j-th column of $\mathbf{0}$, and multiplying it by -2. We then compare it to the different types of matrices in the complement orbit of B.

Case 1: $(i+1,1)$-pivot and $(1,j+1)$-pivot
Let $1 \leqslant i \leqslant m$, by Lemma 3.7.11, we have

$$\operatorname{neg}(A\operatorname{diag}(A_{i+1})) = \begin{bmatrix}\operatorname{neg}(A)_1 \oplus \operatorname{neg}(A)_{i+1}\\ \operatorname{neg}(A)_2 \oplus \operatorname{neg}(A)_{i+1}\\ \vdots\\ \operatorname{neg}(A)_i \oplus \operatorname{neg}(A)_{i+1}\\ \operatorname{neg}(A)_{i+1} \oplus \operatorname{neg}(A)_{i+1}\\ \operatorname{neg}(A)_{i+2} \oplus \operatorname{neg}(A)_{i+1}\\ \vdots\\ \operatorname{neg}(A)_m \oplus \operatorname{neg}(A)_{i+1}\end{bmatrix} = \begin{bmatrix}0 & \mathbf{0}^\top \oplus B_i\\ 0 & B_1 \oplus B_i\\ \vdots & \vdots\\ 0 & B_{i-1}\oplus B_i\\ 0 & \mathbf{0}^\top\\ 0 & B_{i+1}\oplus B_i\\ \vdots & \vdots\\ 0 & B_m \oplus B_i\end{bmatrix},$$

Since $\mathbf{0}^\top \oplus B_i = B_i$, we have $(A\operatorname{diag}(A_{i+1}))/\!\!/(i+1,1) \simeq -2B_{[i]}$, up to a cyclic permutation of the first i rows. By (3.7.4) and $A_{i+1} = 1^\top - 2B_i$ we obtain:

$$A/\!\!/(i+1,1) \simeq -2B_{[i]}\operatorname{diag}(\mathbf{1}^\top - 2B_i) \simeq -2A_{i+1}^1 B_{[i]}\operatorname{diag}(\mathbf{1}^\top - 2B_i).$$

We obtain by similar computations and by (3.7.4) that, for $1 \leqslant j \leqslant n$:

$$A/\!\!/(1,j+1) \simeq -2\operatorname{diag}(\mathbf{1} - 2B^j)B^{[j]} \simeq -2A_1^{j+1}\operatorname{diag}(\mathbf{1} - 2B^j)B^{[j]}.$$

Case 2: $(i+1,j+1)$-pivots
Now, let $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. We first look at how $B_{[i]}^{[j]}$ looks like, depending on B_i^j. Let $J_1 = \operatorname{supp}(B_i) \setminus \{j\}$, $J_0 = \{1,\ldots,m\} \setminus \operatorname{supp}(B_i)$, $I_1 = \operatorname{supp}(B^j) \setminus \{i\}$ and $I_0 = \{1,\ldots,n\} \setminus \operatorname{supp}(B^j)$. Up to putting row i at the top of B and column j to the left, and up to reorganizing the rows and columns, and if $B_i^j = 0$, we may assume that

$$B = \begin{array}{c|c} & \begin{array}{ccc} j & J_1 & J_0\end{array}\\ \begin{array}{c} i\\ I_1\\ I_0\end{array} & \begin{bmatrix}0 & \mathbf{1}^\top & \mathbf{0}^\top\\ \mathbf{1} & B_{I_1}^{J_1} & B_{I_1}^{J_0}\\ \mathbf{0} & B_{I_0}^{J_1} & B_{I_0}^{J_0}\end{bmatrix}\end{array}, \text{ therefore } B_{[i]}^{[j]} = \begin{array}{c|c} & \begin{array}{ccc} j & J_1 & J_0\end{array}\\ \begin{array}{c} i\\ I_1\\ I_0\end{array} & \begin{bmatrix}0 & \mathbf{1}^\top & \mathbf{0}^\top\\ \mathbf{1} & B_{I_1}^{J_1} & \overline{B_{I_1}^{J_0}}\\ \mathbf{0} & \overline{B_{I_0}^{J_1}} & B_{I_0}^{J_0}\end{bmatrix}\end{array}.$$

If $B_i^j = 1$, we have

$$B = \begin{array}{c} \\ i \\ I_1 \\ I_0 \end{array} \begin{array}{c} \begin{array}{ccc} j & J_1 & J_0 \end{array} \\ \begin{bmatrix} 1 & \mathbf{1}^\top & \mathbf{0}^\top \\ \mathbf{1} & B_{I_1}^{J_1} & B_{I_1}^{J_0} \\ \mathbf{0} & B_{I_0}^{J_1} & B_{I_0}^{J_0} \end{bmatrix} \end{array}, \text{ therefore } B_{[i]}^{[j]} = \begin{array}{c} \\ i \\ I_1 \\ I_0 \end{array} \begin{array}{c} \begin{array}{ccc} j & J_1 & J_0 \end{array} \\ \begin{bmatrix} 1 & \mathbf{0}^\top & \mathbf{1}^\top \\ \mathbf{0} & \overline{B_{I_1}^{J_1}} & B_{I_1}^{J_0} \\ \mathbf{1} & B_{I_0}^{J_1} & \overline{B_{I_0}^{J_0}} \end{bmatrix} \end{array}.$$

We now relate the two cases $B_i^j = 0$ and $B_i^j = 1$ with the cases $A_{i+1}^{j+1} = 1$ and $A_{i+1}^{j+1} = -1$, respectively.

Assume $A_{i+1}^{j+1} = 1$, and let

$$\begin{cases} J_- = \operatorname{supp}(\operatorname{neg}(A_{i+1})), \\ J_+ = (\{2, \dots, m+1\} \setminus \operatorname{supp}(\operatorname{neg}(A_{i+1}))) \setminus \{j+1\}, \\ I_- = \operatorname{supp}(\operatorname{neg}(A^{j+1})), \text{ and} \\ I_+ = (\{2, \dots, n+1\} \setminus \operatorname{supp}(\operatorname{neg}(A^{j+1}))) \setminus \{i+1\}. \end{cases}$$

We have, up to reordering the rows and columns:

$$A \simeq \begin{array}{c} \\ 1 \\ I_- \\ i+1 \\ I_+ \end{array} \begin{array}{c} \begin{array}{cccc} 1 & J_- & j+1 & J_+ \end{array} \\ \begin{bmatrix} 1 & \mathbf{1}^\top & 1 & \mathbf{1}^\top \\ \mathbf{1} & A_{I_-}^{J_-} & -\mathbf{1} & A_{I_-}^{J_+} \\ 1 & -\mathbf{1}^\top & 1 & \mathbf{1}^\top \\ \mathbf{1} & A_{I_+}^{J_-} & \mathbf{1} & A_{I_+}^{J_+} \end{bmatrix} \end{array},$$

thus,

$$\operatorname{diag}(A^{j+1}) A \operatorname{diag}(A_{i+1}) \simeq \begin{array}{c} \\ 1 \\ I_- \\ i+1 \\ I_+ \end{array} \begin{array}{c} \begin{array}{cccc} 1 & J_- & j+1 & J_+ \end{array} \\ \begin{bmatrix} 1 & -\mathbf{1}^\top & 1 & \mathbf{1}^\top \\ -\mathbf{1} & A_{I_-}^{J_-} & \mathbf{1} & -A_{I_-}^{J_+} \\ 1 & \mathbf{1}^\top & 1 & \mathbf{1}^\top \\ \mathbf{1} & -A_{I_+}^{J_-} & \mathbf{1} & A_{I_+}^{J_+} \end{bmatrix} \end{array},$$

therefore,

$$\operatorname{neg}(\operatorname{diag}(A^{j+1}) A \operatorname{diag}(A_{i+1})) \simeq \begin{bmatrix} 0 & \mathbf{1}^\top & 0 & \mathbf{0}^\top \\ \mathbf{1} & B_{I_1}^{J_1} & \mathbf{0} & \overline{B_{I_1}^{J_0}} \\ 0 & \mathbf{0}^\top & 0 & \mathbf{0} \\ \mathbf{0} & \overline{B_{I_0}^{J_1}} & \mathbf{0} & B_{I_0}^{J_0} \end{bmatrix},$$

with $J_1 = \{j-1\colon j \in J_-\}$, $J_0 = \{j-1\colon j \in J_+\}$, $I_1 = \{i-1\colon i \in I_-\}$, and $I_0 = \{i-1\colon i \in I_+\}$. This yields, $\operatorname{diag}(A^{j+1})A\operatorname{diag}(A_{i+1})/\!\!/(i+1,j+1) \simeq -2B_{[i]}^{[j]}$, up to cyclic permutation of the first i rows and j columns. Finally, by (3.7.4)

$$\begin{aligned} A/\!\!/(i+1,j+1) &\simeq -2\operatorname{diag}(\mathbf{1}-2B^j)B_{[i]}^{[j]}\operatorname{diag}(\mathbf{1}-2B_i) \\ &\simeq -2A_{i+1}^{j+1}\operatorname{diag}(\mathbf{1}-2B^j)B_{[i]}^{[j]}\operatorname{diag}(\mathbf{1}-2B_i). \end{aligned}$$

Assume $A_{i+1}^{j+1} = -1$, and let

$$\begin{cases} J_- = \operatorname{supp}(\operatorname{neg}(A_{i+1})) \setminus \{j+1\}, \\ J_+ = \{2,\dots,m+1\} \setminus \operatorname{supp}(\operatorname{neg}(A_{i+1})), \\ I_- = \operatorname{supp}(\operatorname{neg}(A^{j+1})) \setminus \{i+1\}, \text{ and} \\ I_+ = \{2,\dots,n+1\} \setminus \operatorname{supp}(\operatorname{neg}(A^{j+1})). \end{cases}$$

We have:

$$A = \begin{array}{c} \\ 1 \\ I_- \\ i+1 \\ I_+ \end{array} \begin{array}{c} \begin{array}{cccc} 1 & J_- & j+1 & J_+ \end{array} \\ \begin{bmatrix} 1 & \mathbf{1}^\top & 1 & \mathbf{1}^\top \\ \mathbf{1} & A_{I_-}^{J_-} & -\mathbf{1} & A_{I_-}^{J_+} \\ 1 & -\mathbf{1}^\top & -1 & \mathbf{1}^\top \\ \mathbf{1} & A_{I_+}^{J_-} & \mathbf{1} & A_{I_+}^{J_+} \end{bmatrix} \end{array}.$$

Multiplying by $\operatorname{diag}(-A^{j+1})$ and $\operatorname{diag}(A_{i+1})$, we obtain the following:

$$\operatorname{diag}(-A^{j+1})A\operatorname{diag}(A_{i+1}) = \begin{array}{c} \\ 1 \\ I_- \\ i+1 \\ I_+ \end{array} \begin{array}{c} \begin{array}{cccc} 1 & J_- & j+1 & J_+ \end{array} \\ \begin{bmatrix} -1 & \mathbf{1}^\top & 1 & -\mathbf{1}^\top \\ \mathbf{1} & -A_{I_-}^{J_-} & \mathbf{1} & A_{I_-}^{J_+} \\ 1 & \mathbf{1}^\top & 1 & \mathbf{1}^\top \\ -\mathbf{1} & A_{I_+}^{J_-} & \mathbf{1} & -A_{I_+}^{J_+} \end{bmatrix} \end{array},$$

therefore

$$\operatorname{neg}(\operatorname{diag}(-A^{j+1})A\operatorname{diag}(A_{i+1})) \simeq \begin{bmatrix} 1 & \mathbf{0}^\top & 0 & \mathbf{1}^\top \\ \mathbf{0} & \overline{B_{I_1}^{J_1}} & \mathbf{0} & B_{I_1}^{J_0} \\ 0 & \mathbf{0}^\top & 0 & \mathbf{0} \\ \mathbf{1} & B_{I_0}^{J_1} & \mathbf{0} & \overline{B_{I_0}^{J_0}} \end{bmatrix},$$

with $J_1 = \{j-1\colon j \in J_-\}$, $J_0 = \{j-1\colon j \in J_+\}$, $I_1 = \{i-1\colon i \in I_-\}$, $I_0 = \{i-1\colon i \in I_+\}$. This yields, $\operatorname{diag}(-A^{j+1})A\operatorname{diag}(A_{i+1})/\!\!/(i+1,j+1) \simeq -2B_{[i]}^{[j]}$, up to cyclic permutation of the first i rows and j columns. Finally, by (3.7.4)

$$\begin{aligned} A/\!\!/(i+1,j+1) &\simeq 2\operatorname{diag}(\mathbf{1}-2B^j)B_{[i]}^{[j]}\operatorname{diag}(\mathbf{1}-2B_i) \\ &\simeq -2A_{i+1}^{j+1}\operatorname{diag}(\mathbf{1}-2B^j)B_{[i]}^{[j]}\operatorname{diag}(\mathbf{1}-2B_i). \end{aligned}$$

$\square$

Classification of te-interlaces

This section is devoted to the proof of the following.

Lemma 3.7.14. *Only two types of te-interlaces exist: the thin and the thick ones.*

Proof. Let A be a te-interlace. Since the rows of A all have the same support, up to restricting to a subset of columns of A, we may assume that A is ± 1. We write

$$A = \begin{bmatrix} 1 & \mathbf{1}^\top \\ \mathbf{1} & \mathbf{J} - 2B \end{bmatrix},$$

for B a core of A. By Theorem 3.5.3, all the trims of A are totally equimodular. Since the complement orbit of B is composed of resigning of these trims by Lemma 3.7.12, B and all the matrices in its complement orbit are totally equimodular 0,1 matrices. Moreover, $\mathrm{eqdet}(A) = 2^{n-1}\,\mathrm{eqdet}(B_{[i]}^{[j]})$ for every pivot $(i+1, j+1)$, hence $\mathrm{eqdet}(B_{[i]}^{[j]}) = \mathrm{eqdet}(B)$.

If $\mathrm{eqdet}(B) = 1$, then B is totally unimodular, as well as all the $B_{[i]}^{[j]}$'s, since B and its complements are all totally equimodular. Therefore B is complement totally unimodular. Moreover, $\mathrm{eqdet}(A) = 2^{n-1}\,\mathrm{eqdet}(B) = 2^{n-1}$. Hence, A is a thin te-interlace.

We now show that if $\mathrm{eqdet}(B) \geqslant 2$, then A is a thick te-interlace. Then, by Lemma 3.7.9, B contains a te-lace D. Let M be a $|D| \times |D|$ invertible submatrix of D. Note that M is minimally non-totally unimodular.

Claim 3.7.15. *M is complement minimally non-totally unimodular.*

Proof. Since A is totally equimodular, its submatrix

$$A' = \begin{bmatrix} 1 & \mathbf{1}^\top \\ \mathbf{1} & \mathbf{J} - 2M \end{bmatrix},$$

is totally equimodular. Again, Lemmas 3.7.3 and 3.7.12 imply that the complement orbit of M contains only totally equimodular matrices of determinant $\det(M) = \pm 2$. Since proper submatrices of M are totally unimodular, removing a row at index $s > 1$ and a column at index $d > 1$ of A', and then applying Lemmas 3.7.3 and 3.7.12 yields a totally unimodular matrix. All the proper submatrices of M are therefore complement totally unimodular. Therefore, M is complement minimally non-totally unimodular, as desired. $\square$

Moreover, M is of odd size by Theorem 3.7.10. By Lemma 3.7.8 and since D is 0,1, the nonzero columns of D are copies of those of M.

Claim 3.7.16. *B is a te-lace.*

Proof. Suppose not, then $D \subsetneq B$, and let $r \in B \setminus D$. There exists a $(|D|+1) \times (|D|+1)$ totally equimodular invertible submatrix N of $D \cup \{r\}$ that we write:

$$N = \begin{bmatrix} t & w^\top \\ v & M \end{bmatrix},$$

where M, the scalar t, and the vectors v and w are 0,1, and M is complement minimally non-totally unimodular by Claim 3.7.15. Since N is a submatrix of B whose complement orbit contains only totally equimodular $0,1$ matrices, every matrix in the complement orbit of N is totally equimodular.

Fact. We may assume $v = \mathbf{0}$ and $t = 1$.

By Lemma 3.7.8, since N is totally equimodular and 0,1, v is either 0 or M^j, for some j. Suppose $v = M^j$. Since N is invertible, this implies that t and the coefficient w_j above M^j are distinct, and up to permutation of these two columns, suppose $t = 1$. Therefore, N and its column-$(j+1)$ complement are as follows:

$$N = \begin{bmatrix} 1 & w_{<j}^\top & 0 & w_{>j}^\top \\ M^j & M^{<j} & M^j & M^{>j} \end{bmatrix} \text{ and } N^{[j+1]} = \begin{bmatrix} 1 & w^\top \\ \mathbf{0} & M^{[j]} \end{bmatrix}.$$

Moreover, $M^{[j]}$ is minimally non-totally unimodular by Claim 3.7.15. We are therefore in the case $v = \mathbf{0}$, as desired.

Fact. We may assume $w = \mathbf{0}$.

We have

$$N = \begin{bmatrix} 1 & w^\top \\ \mathbf{0} & M \end{bmatrix} \text{ and } (N^{-1})^\top = \begin{bmatrix} 1 & \mathbf{0}^\top \\ z & (M^{-1})^\top \end{bmatrix},$$

where $z = -(M^{-1})^\top w$. Up to replacing N by $N^{[1]}$, which changes w to $w' = 2 \cdot \mathbf{1} - w$, having complementary support to w, and since w is of odd size, we may assume that $\text{supp}(w)$ is even. By Theorem 3.7.7, M^{-1} has only $\pm\frac{1}{2}$ entries, and hence $z = -(M^{-1})^\top w$ is integer. If $w \neq \mathbf{0}$, then $z \neq \mathbf{0}$ since M is invertible. Then, any row of $(N^{-1})^\top$ indexed in $\text{supp}(-(M^{-1})^\top w)$ contains a nonzero integer in its first column and $\pm\frac{1}{2}$ elsewhere. Such a row is not equimodular, contradicting the total equimodularity of $(N^{-1})^\top$ given by that of N and Theorem 3.5.5.

We now end the proof of Claim 3.7.16. By the two above facts, we may assume

$$N = \begin{bmatrix} 1 & \mathbf{0}^\top \\ \mathbf{0} & M \end{bmatrix} \text{ and therefore } N_{[1]} = \begin{bmatrix} 1 & \mathbf{0}^\top \\ \mathbf{1} & M \end{bmatrix}.$$

By Theorem 3.7.7, since M is 0,1 and minimally non-totally unimodular, it has an even number of 1's in each column. As M is of odd size, $\mathbf{1}$ is not a column of M. Therefore, by Lemma 3.7.8, $N_{[1]}$ is not totally equimodular, a contradiction since $N_{[1]}$ is in the complement orbit of N. Thus, B is a te-lace. □

Therefore, $\text{eqdet}(A) = \pm 2^{n-1} \text{eqdet}(B) = \pm 2^n$, and hence A is a thick te-interlace. □

Consequences

Lemma 3.7.14 and the interplay between te-interlaces and complement matrices have several consequences. First, combined with Lemma 3.7.12, this gives the following.

Corollary 3.7.17. *Let p be a pivot of a te-interlace A. Then, we have:*

— *A is thin if and only if $\widehat{A /\!\!/ p}$ is a tu-set,*

— *A is thick if and only if $\widehat{A /\!\!/ p}$ is a te-lace.*

In particular, the second point of Corollary 3.7.17 combined with Lemma 3.7.8 yield the following.

Corollary 3.7.18. *Let A' be an $n \times n$ invertible submatrix of a thick te-interlace A of size n. Then, each nonzero column of A is a column of A' or its opposite.*

Corollary 3.7.17 can be strengthened as follows, in the case of square matrices.

Corollary 3.7.19. *Let A be a ± 1 square invertible matrix and B a core of A. Then, the following holds:*

- *A is a thin te-interlace if and only if B is complement totally unimodular,*
- *A is a thick te-interlace if and only if B is complement minimally non-totally unimodular.*

Proof. For both cases, the "only if" part comes from the proof of Lemma 3.7.14. To see the "if" part, first note that, up to resigning, the trims of A are in the complement orbit of B by Lemma 3.7.12. Since B is either complement totally unimodular or complement minimally non-totally unimodular, all these trims are either totally unimodular or minimally non-totally unimodular, hence are all totally equimodular. Therefore, so is A by Theorem 3.5.2. Since A is ± 1, it is a te-interlace, and a determinant computation concludes each case. ☐

In particular, since proper minors of complement minimally non-totally unimodular matrices are complement totally unimodular matrices, we have the following.

Remark 3.7.20. Let A be a te-interlace. If A is thin, then $\operatorname{eqdet}(B) = 2^{|B|-1}$, for every $B \subseteq A$. If A is thick, then $\operatorname{eqdet}(B) = 2^{|B|-1}$, for every $B \subsetneq A$, and $\operatorname{eqdet}(A) = 2^{|A|}$.

Moreover, we also obtain the following.

Corollary 3.7.21. *For a square invertible ± 1 matrix A, the following statements are equivalent:*

— *A is totally equimodular,*

— *$A^\top$ is totally equimodular,*

— *A^{-1} is totally equimodular,*

— *$(A^{-1})^\top$ is totally equimodular.*

Proof. Square invertible totally equimodular ± 1 matrices are te-interlaces. By Lemma 3.7.14, if A is totally equimodular then it is either a thick or a thin te-interlace. In both cases, by Remark 3.7.20 the $k \times k$ determinants of A are either 0 or $\pm 2^{k-1}$, for $k < n$. Hence, A is totally equimodular if and only $A^\top$ is, and Theorem 3.5.5 concludes. ☐

Using the comatrix and determinant computations yield the following.

Corollary 3.7.22. *Let A be a square ± 1 matrix, then*

— *A is a thin te-interlace if and only if $\left(\frac{1}{2}A\right)^{-1}$ is a te-lace,*

— *A is a thick te-interlace if and only if $\left(\frac{1}{4}A\right)^{-1}$ is a thick te-interlace.*

The first statement means that, up to rescaling, the inverse of minimally non-totally unimodular matrices are the thin te-interlaces.

3.7.3 Proof of Lemma 3.5.7

Note that this section is different from the one in the preprint [35], we provide here a newer proof that leverages the theory of binary matroids.

A crucial result to decompose te-sets is the following, which can be restated as: if two distinct te-laces of a te-set intersect, then their union is a subset of a te-interlace.

In brief, the proof is constructed by considering a counterexample with a minimum number of rows. The impact of well-chosen trims is then examined. The analysis yields a set of properties for such a counterexample, in particular on how the trims modify the supports of the rows of the matrix, ultimately leading to a contradiction of the linear independence.

Let $\mathbb{F} = \mathbb{Q}$, $\mathbb{Z}_2$, or $\mathbb{Z}_3$, be the field of rational numbers, the mod 2 field $\mathbb{Z}_2$ whose elements are $\{0, 1\}$, or the mod 3 field $\mathbb{Z}_3$ with elements $\{0, \pm 1\}$. Given a set of integer vectors $A = \{a_1, \ldots, a_m\} \subseteq \mathbb{Z}^n$, its associated *(abstract)* $\mathbb{F}$*-matroid* $M(A; \mathbb{F})$ is the subset of $\mathcal{P}(\{1, \ldots, m\})$ containing every subsets $I \subseteq \{1, \ldots, m\}$ such that $\{(a_i)_\mathbb{F}\}_{i \in I}$ are $\mathbb{F}$-independent, with

$$x_\mathbb{F} = \begin{cases} x \text{ if } \mathbb{F} = \mathbb{Q}, \\ x \pmod 2 \text{ if } \mathbb{F} = \mathbb{Z}_2, \\ x \pmod 3 \text{ if } \mathbb{F} = \mathbb{Z}_3. \end{cases}$$

We call $M(A; \mathbb{Z}_2)$ the *binary matroid* associated to A and $M(A, \mathbb{Z}_3)$ the *ternary matroid* associated to A. The integer m is called the *size* of the matroid. A $\mathbb{Z}_2$-circuit of $M(A; \mathbb{Z}_2)$ is a subset $C \subseteq \{1, \ldots, m\}$ such that $C \notin M(A; \mathbb{Z}_2)$ but every proper subset X of C is in $M(A, \mathbb{Z}_2)$.

A matroid or rank r is *regular* if it can be represented over any field and in particular, by Tutte [134] it is regular if it can be represented over $\mathbb{Z}_2$ and $\mathbb{Z}_3$ by a matrix of row rank r. By Gerards [77], since a binary matroid is represented by a 0,1 matrix of row rank r, the matroid is regular if and only if some 1's can be changed to -1's such that the obtained matrix is totally unimodular.

A binary matroid M is *te-representable* if there exists a te-set A such that $M = M(A; \mathbb{Z}_2)$ and $M(A, \mathbb{Z}_3) = \mathcal{P}(\{1, \ldots, m\})$.

Lemma 3.7.23. *Let A be a te-set of size m. Then,*

1. $M(A; \mathbb{Z}_3) = M(A; \mathbb{Q}) = \mathcal{P}(\{1, \ldots, m\})$.
2. *if A is a tu-set, then $M(A; \mathbb{Z}_2) = \mathcal{P}(\{1, \ldots, m\})$, and A is a regular matroid;*
3. *if A is a te-lace, then $M(A; \mathbb{Z}_2) = \mathcal{P}(\{1, \ldots, m\}) \setminus \{1, \ldots, m\}$, notably, $C = \{1, \ldots, m\}$ is a $\mathbb{Z}_2$-circuit;*
4. *if A is a te-interlace, then $M(A; \mathbb{Z}_2) = \{\{1\}, \ldots, \{m\}\}$, notably, every $C \subseteq \{1, \ldots, m\}$ with $|C| = 2$ is a $\mathbb{Z}_2$-circuit.*

In other words, te-laces correspond to the $\mathbb{Z}_2$-circuits of the binary matroid associated to A. Moreover, A has no $\mathbb{Z}_3$-circuits.

The *restriction* of a matroid on $\{1, \ldots, m\}$ to a subset $X \subseteq \{1, \ldots, m\}$ is the matroid defined by

$$M|_X = \{I \cap X : I \in M\},$$

Note that the restriction preserves the $\mathbb{F}$-representability, and therefore the te-representability.

The *contraction* of k in a matroid M on $\{1, \ldots, m\}$ is

$$M/k = \{I \in M : I \not\ni k, I \cup \{k\} \in M\},$$

on $\{1, \ldots, m\} \setminus \{k\}$.

Lemma 3.7.24. *Let A be a te-set, and let $a_k \in A$ being in no te-lace of size 2. Then for any pivot $p = (k, j)$, we have $M(A /\!\!/ p, \mathbb{F}) = M(A, \mathbb{F})/k$. In particular, if M is te-representable, then M/k is te-representable for every k not contained in any $\mathbb{Z}_2$-circuit of size 2. Moreover, if C is a $\mathbb{Z}_2$-circuit of size at least 3 containing k, then $C \setminus \{k\}$ is a $\mathbb{Z}_2$-circuit of $M(A, \mathbb{F})/k$.*

We will use the following characterization of binary matroids.

Theorem 3.7.25 ([112])**.** *A matroid M is binary if and only if for any two circuits of M, their symmetric difference is a disjoint union of circuit.*

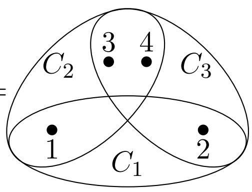
We are now ready to prove Lemma 3.5.7.

Lemma 3.7.26 (Lemma 3.5.7)**.** *In a te-set, if two distinct te-laces intersect, then they are both of size two and their symmetric difference is also a te-lace.*

Proof. The smallest counterexample we can think of is a binary matroid $\underline{M} = C_1 \cup C_2$ of size 4, for two $\mathbb{Z}_2$-circuits C_1 and C_2, with $|C_1| = 2$, $|C_2| = 3$ and $|C_1 \cap C_2| = 1$. We show in Claim 3.7.27 that it is not te-representable. Then, we show in Claim 3.7.28 that any bigger counterexample can be turned into $\underline{M}$ by mean of operations preserving the te-representability, a contradiction.

Claim 3.7.27. *Let $\underline{M} = C_1 \cup C_2$ be a binary matroid on 4 elements, for two $\mathbb{Z}_2$-circuits C_1 and C_2, with $|C_1| = 2$, $|C_2| = 3$ and $|C_1 \cap C_2| = 1$. Then $\underline{M}$ is not te-representable.*

Proof. Suppose that $\underline{M}$ is te-representable, so we are given a 0,±1 set of vectors $\underline{A}$ such that $\underline{M} = M(\underline{A}; \mathbb{Z}_2)$ and $M(\underline{A}; \mathbb{Z}_3) = \mathcal{P}(\{1, 2, 3, 4\})$. Say $C_1 = \{1, 2\}$ and $C_2 = \{1, 3, 4\}$, then by Theorem 3.7.25, $C_1 \triangle C_2 = \{2, 3, 4\}$ is a disjoint union of $\mathbb{Z}_2$-circuits. If there is a singleton $\{i\} \subseteq \{2, 3, 4\}$ which is a $\mathbb{Z}_2$-circuit, this means the vector a_i has even entries, so is zero since $\underline{A}$ is 0,±1. But this means that $\{i\}$ is also a $\mathbb{Z}_3$-circuit, which contradicts the te-representability. If there is a subset $\{i, j\} \subseteq \{2, 3, 4\}$ of size 2 which is a $\mathbb{Z}_2$-circuit, then $\{2, 3, 4\} \setminus \{i, j\}$ is a singleton $\mathbb{Z}_2$-circuit, and we are back to the previous case. Therefore, $C_3 = \{2, 3, 4\}$ is a $\mathbb{Z}_2$-circuit of $\underline{M}$. The picture is as follows:

$\underline{M} =$ 

Since $\{a_1, a_2, a_3, a_4\}$ is $\mathbb{Z}_3$-independent, there exists a 4×4 submatrix A' of A which is invertible and of the form

$$A' = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} a_1^1 & a_1^2 & a_1^3 & a_1^4 \\ 1 & 1 & 0 & v_1 \\ 1 & 0 & 1 & v_2 \\ 0 & 1 & 1 & v_3 \end{bmatrix},$$

up to resigning the rows and columns, for $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ being $\mathbf{0}$ or the copy of one of the columns on its left, by Lemma 3.7.8 since $A_{\{2,3,4\}}$ is a te-lace. Since $\{1, 2\}$ is a $\mathbb{Z}_2$-circuit, we have $\mathrm{supp}(a_1) = \mathrm{supp}(a_2)$, and therefore:

$$A' = \begin{bmatrix} \pm 1 & \pm 1 & 0 & \pm v_1 \\ 1 & 1 & 0 & v_1 \\ 1 & 0 & 1 & v_2 \\ 0 & 1 & 1 & v_3 \end{bmatrix}.$$

But $v \neq \mathbf{0}$, as otherwise, A' is not invertible. Similarly, $v_1 \neq 0$, as otherwise, A' has twice the same column and is not invertible. Up to permuting the last two rows and first two columns of A', we may assume that

$$A' = \begin{bmatrix} \pm 1 & \pm 1 & 0 & \pm 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

For A' to be invertible, up to resigning the first row or exchanging the first and last columns, we need

$$A' = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

Now, deleting the second and last rows yields

$$A'_{\{1,3\}} = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 1 \end{bmatrix},$$

which is not totally equimodular.

Finally, $\underline{M}$ is not te-representable. □

Claim 3.7.28. *A te-representable binary matroid M of size $m \geqslant 5$, having two $\mathbb{Z}_2$-circuits C_1 and C_2 where $|C_1 \cup C_2| = m$, with $|C_1| = 2$ and $|C_2| \geqslant 4$, or $|C_1| \geqslant 3$ and $|C_2| \geqslant 3$, can be turned into $\underline{M}$ by mean of operations preserving the te-representability.*

Proof. We first prove the lemma for the case where $|C_1| = 2$ and $|C_2| \geqslant 4$. In that case, we have $|C_1 \cap C_2| = 1$, as otherwise $C_1 \subsetneq C_2$, which is impossible by definition of a $\mathbb{Z}_2$-circuit. Then, by Theorem 3.7.25, the symmetric difference $C_1 \triangle C_2$ is a disjoint union of $\mathbb{Z}_2$-circuits of M. Only one of them covers the single element in $C_2 \setminus C_1$, the rest are $\mathbb{Z}_2$-circuits D strictly included in C_2, which is impossible by definition. Hence $C_3 = C_1 \triangle C_2$ is a $\mathbb{Z}_2$-circuit of size $|C_2| \geqslant 4$. Moreover, C_1, C_2, and C_3 are the only $\mathbb{Z}_2$-circuits of M. Hence any element $k \in C_2 \cap C_3$ is contained in no $\mathbb{Z}_2$-circuits of size 2, hence the contraction of M at k is of size $m-1$ and is a smaller te-representable counterexample by 3.7.24. Repeating this process eventually leads to $\underline{M}$ since it decreases m by 1 at each step.

We finally prove that when $|C_1| \geqslant 3$ and $|C_2| \geqslant 3$, then we can construct a strictly smaller te-representable counterexample. Firstly, we may assume that M contains no $\mathbb{Z}_2$-circuits C of size 2, as otherwise, one among C_1 and C_2 will intersect it and we are back to the previous case by restricting to $C \cup C_i$, for $i = 1$ or 2. Suppose that $|C_1 \cap C_2| \geqslant 2$. Every $k \in C_1 \cap C_2$ is contained in no $\mathbb{Z}_2$-circuit of size 2 and therefore, the contraction of M at k yields a smaller te-representable counterexample bya smaller Lemma 3.7.24. So C_1 and C_2 intersect at a single element, and M is of size at least 5. Once more, by Theorem 3.7.25, the symmetric difference $C_1 \triangle C_2 = \bigsqcup_{i=1}^{k} D_i$ is a disjoint union of $\mathbb{Z}_2$-circuits of M. As previously, every such $\mathbb{Z}_2$-circuits C has a nonempty intersection with both C_1 and C_2, by definition. Say $D_1 = C \cap C_1$ and $D_2 = C \cap C_2$, so $C = D_1 \sqcup D_2$, with $|D_1| < |C_1|$ and $|D_2| < |C_2|$. Since $|C| \geqslant 3$, one among D_1 and D_2 has size at least 2, say D_1 without loss of generality. We have $4 \leqslant |C \cup C_1| \leqslant |C_1 \cup C_2|$. If $|C \cup C_1| = 4$, this means that $|C \triangle C_1$ is a $\mathbb{Z}_2$-circuit of size 2, a contradiction. Hence, the restriction of M on $C \cup C_1$ followed by its contraction at any k in D_1 is of size $\geqslant 4$ and yields a strictly smaller te-representable counterexample, by Lemma 3.7.24. Repeating this process yields one among C_1 or C_2 being of size 2, with the other one remaining of size $\geqslant 3$ and we eventually reach the previous case or $\underline{M}$. □

□

3.7.4 Proof of the decomposition theorem

Thank to the results of the previous sections, we are now in position to prove the decomposition theorem.

Recall that disjoint te-bricks are mutually-tu when the sets which intersect several of them while containing none of the te-laces and at most one vector of each te-interlace are tu-sets.

Theorem 3.7.29 (Theorem 3.5.6)**.** *A linearly independent set of* $0,\pm 1$ *vectors* A *is a te-set if and only if it is the disjoint union of mutually-tu te-bricks. More precisely:*

$$A = \underbrace{U}_{\textit{tu-set}} \sqcup \underbrace{L_1 \sqcup \cdots \sqcup L_k}_{\textit{te-laces}} \sqcup \underbrace{S_1 \sqcup \cdots \sqcup S_s}_{\textit{thin te-interlaces}} \sqcup \underbrace{T_1 \sqcup \cdots \sqcup T_t}_{\textit{thick te-interlaces}}.$$

Proof. Let us denote the property on the right by $(*)$. Note that the property of being mutually-tu implies that the decomposition is unique, up to considering te-laces of size two either as te-interlaces or as te-laces. In this proof, we will make no distinction between thin and thick te-interlaces. In fact, once we obtain a decomposition of the form

$$A = \underbrace{U}_{\text{tu-set}} \sqcup \underbrace{L_1 \sqcup \cdots \sqcup L_k}_{\text{te-laces}} \sqcup \underbrace{I_1 \sqcup \cdots \sqcup I_s}_{\text{te-interlaces}}$$

we use Lemma 3.7.14 to obtain that the I_i's are either thin or thick.

The fact that te-sets satisfy $(*)$ comes from Lemma 3.7.26. Indeed, let A be a te-set and $\mathcal{L}$ its family of te-laces. By Lemma 3.7.9, $U = A \setminus (\bigcup_{L \in \mathcal{L}} L)$ is a tu-set. By Lemma 3.7.26, intersecting te-laces are part of a te-interlace. Thus, the intersecting sets of $\mathcal{L}$ can be regrouped into disjoint maximal te-interlaces. Let $\mathcal{I}$ denote this family of te-interlaces. The remaining te-laces $\mathcal{L}'$ of $\mathcal{L}$ are disjoint. Note that $\{U\} \cup \mathcal{L}' \cup \mathcal{I}$ is a partition of A. By construction, no te-lace intersects distinct sets among, U, $L \in \mathcal{L}'$, and $I \in \mathcal{I}$, thus these sets are mutually-tu, and A satisfies $(*)$.

To see the converse, we will need the following claims, where A is a full row rank matrix satisfying $(*)$, $B \subsetneq A$ denotes a subset of rows of A, and ℓ is a row of A which is not in B.

First remark that if $\ell \in A$ has $|\operatorname{supp}(\ell)| = 1$, then $A \setminus \{\ell\}$ is a te-set if and only if A is a te-set.

Claim 3.7.30. *If* $B \subseteq A$*, then* B *satisfies* $(*)$*.*

Proof. Denote by $\mathcal{U} \cup \mathcal{L} \cup \mathcal{I}$ the decomposition of the rows of A given by $(*)$, where $\mathcal{U} = \{U\}$ and U is a tu-set, every $L \in \mathcal{L}$ is a te-lace, and every $I \in \mathcal{I}$ is a maximal te-interlace. Let $\mathcal{L}_B = \{L : \text{ for } L \in \mathcal{L} \text{ with } L \subseteq B\}$, $\mathcal{I}_B = \{I \cap B : \text{ for } I \in \mathcal{I} \text{ with } |I \cap B| \geqslant 2\}$, and $\mathcal{U}_B = \{U_B\}$, where U_B is the union of $U \cap B$, $L \cap B$ for $L \in \mathcal{L}$ with $L \setminus B \neq L$, and $I \cap B$ for $I \in \mathcal{I}$ with $|I \cap B| = 1$.

Note that $\mathcal{U}_B \cup \mathcal{L}_B \cup \mathcal{I}_B$ is a partition of B. By construction, U_B is a tu-set since the te-bricks of A are mutually-tu. Each set of $\mathcal{L}_B$ is a te-lace, and each set of $\mathcal{I}_B$ is a te-interlace. Now, the sets of this partition are mutually-tu because $B \subseteq A$ and A satisfies $(*)$.

Therefore, B satisfies $(*)$. □

Claim 3.7.31. *If* B *is a te-lace contained in no te-interlace and* $\ell \in A \setminus B$*, then* $(B \cup \{\ell\}) /\!\!/ (\ell, j)$ *is a te-lace for all* $j \in \operatorname{supp}(\ell)$*.*

Proof. Let $p = (\ell, j)$ be a pivot of A.

We first prove that $B \cup \{\ell\}$ is a te-set by induction on its number of rows. Since the te-bricks of A are mutually-tu, $B \setminus \{b\} \cup \{\ell\}$ is a tu-set for all $b \in B$. Moreover, B is a te-lace. Therefore, all there is to show is that $B \cup \{\ell\}$ is equimodular.

If $B = \{b_1, b_2\}$ has size two, then the result holds since $\operatorname{supp}(b_1) = \operatorname{supp}(b_2)$ and both $\{\ell, b_1\}$ and $\{\ell, b_2\}$ are tu-sets by $(*)$. Thus, assume $|B| \geqslant 3$ and let $q = (b, j')$ be a pivot of $B \cup \{\ell\}$ for some $b \in B$. By Lemma 3.7.4 and since trimming preserves tu-sets, $(B \cup \{\ell\}) /\!\!/ q$ is the disjoint union of the te-lace $B /\!\!/ q$ and the row obtained from ℓ by q-trimming, and they are mutually-tu. In particular, the associated matrix is equimodular by induction. Since this holds for all $j' \in \operatorname{supp}(b)$, $B \cup \{\ell\}$ is equimodular by Lemma 3.7.2.

Therefore, $B \cup \{\ell\}$ is a te-set, and hence so is $B' = (B \cup \{\ell\}) /\!\!/ p$ by Theorem 3.7.3. Moreover, B' is not a tu-set as $B \cup \{\ell\}$ is not a tu-set. The fact that B' is a te-lace then follows, because all its proper subsets are tu. Indeed, $B \setminus \{b\} \cup \{\ell\}$ is a tu-set for all $b \in B$, and tu-sets are preserved by trimming. Hence, if b' denotes the row obtained from b by p-trimming, $B' \setminus \{b'\}$ is a tu-set for all $b' \in B'$. □

Claim 3.7.32. *If B is a maximal te-interlace of A and $\ell \in A \setminus B$, then $B' = (B \cup \{\ell\}) /\!\!/ (\ell, j)$ is a maximal te-interlace of $A /\!\!/ (\ell, j)$ for all $j \in \mathrm{supp}(\ell)$. Moreover,* $\mathrm{eqdet}(B') = \mathrm{eqdet}(B)$.

Proof. Let $p = (\ell, j)$ be a pivot of A, $B' = (B \cup \{\ell\}) /\!\!/ p$, and $A' = A /\!\!/ p$. We may assume that the p-pivot modifies B, as otherwise B' is obtained from B by removing a column of zeros and the result holds. Recall that all the rows of B have the same support, which we denote by $\mathrm{supp}(B)$. Since each pair of rows of B is a te-lace, they remain te-laces after p-trimming by Claim 3.7.31, thus B' is not a tu-set, and each pair of rows of B' forms a te-lace.

Let us prove that B' is a te-set [6] and $\mathrm{eqdet}(B') = \mathrm{eqdet}(B)$. Since $\{\ell, b\}$ is a tu-set, ℓ and b coincide on their common support (up to multiplying b by -1), for all $b \in B$. Therefore B' is composed of B on $\mathrm{supp}(B) \setminus \mathrm{supp}(\ell)$, of $-\ell$ on $\mathrm{supp}(\ell) \setminus \mathrm{supp}(B)$, and of zeros elsewhere. The new nonzero columns indexed at $\mathrm{supp}(\ell) \setminus \mathrm{supp}(B)$ are either equal or opposite to the column of B on which the p-pivot occured. Since B is a te-interlace, it is a te-set, and hence B' is a te-set. This also shows that all the nonzero maximal determinants of B and B' have the same absolute value.

Therefore, B' is a te-interlace, and all that remains to prove is that it is a maximal te-interlace of A', that is, no row of $A' \setminus B'$ has $\mathrm{supp}(B')$ as support. By contradiction, suppose not and let s' be a row of $A' \setminus B'$ with $\mathrm{supp}(s') = \mathrm{supp}(r')$ for some $r' \in B'$, let s be the corresponding row of $A \setminus B$. Lemma 3.7.6 applies as $\{\ell, r\}$ is a tu-set, hence either $\{r, s\}$ os $\{\ell, r, s\}$ is a te-lace. The first possibility is impossible because B is maximal and $s \notin B$, thus $\{r, s\}$ is a tu-set. Neither is the other possibility, as it contradicts the fact that the te-bricks of A are mutually-tu. □

Now, we proceed by induction on the cardinality of the set, that is, on the number of rows of the associated matrix. The case of a singleton is immediate. Assume that A satisfies $(*)$, and that all full row rank sets with fewer elements that satisfy $(*)$ are te-sets. Then, all there is to prove is that A is equimodular. We will trim A while preserving $(*)$, hence find equimodular matrices by the induction hypothesis. Then, we will retrieve the equimodularity of A using Lemma 3.7.2.

Denote by $\{U\} \cup \mathcal{L} \cup \mathcal{I}$ the decomposition of the rows of A given by $(*)$, where U is a tu-set, each $L \in \mathcal{L}$ is a te-lace, and each $I \in \mathcal{I}$ is a te-interlace, and these sets are disjoint and mutually-tu. We assume that $|L| \geqslant 3$ for all $L \in \mathcal{L}$ by considering possible te-laces of size 2 of $\mathcal{L}$ as te-interlaces. Note that each of U, $\mathcal{L}$, and $\mathcal{I}$ can be empty. Moreover, the full row rank of A together with $(*)$ imply that each $I \in \mathcal{I}$ is a maximal te-interlace of A. In particular, for each $I \in \mathcal{I}$ and $\ell \in A \setminus I$, the support of ℓ differs from that of the rows of I.

The proof is divided in two cases.

Case 1. A contains a row ℓ which is in no te-interlace.

This case handled in the following claim.

Claim 3.7.33. *Then, $A /\!\!/ p$ satisfies $(*)$ for all pivots $p = (\ell, j)$ of A.*

6. We mention a subtlety here: in any full row rank ± 1 matrix, each pair of rows forms a te-lace. However, there are ± 1 matrices which are not te-sets, hence not te-interlaces.

Proof. Let $p = (\ell, j)$ be a pivot of A. Recall that ℓ belongs either to U, or to some te-lace L_ℓ of $\mathcal{L}$. We treat both cases simultaneously. The two previous claims imply that the te-lace and te-interlaces of A and $A' = A/\!\!/p$ are the same sets of rows, except possibly for L_ℓ if it exists.

When $\ell \in L_\ell$ of $\mathcal{L}$, we have $|L_\ell| \geqslant 3$ since ℓ is in no te-interlace, hence $L'_\ell = L_\ell/\!\!/p$ is a te-lace of A' by Lemma 3.7.4. Moreover, either $\ell \in U$, or $U \cup \{\ell\}$ is a tu-set by the mutually-tu property in $(*)$. In both cases, $U' = (U \cup \{\ell\})/\!\!/p$ is a tu-set by Theorem 3.5.1. Let $\mathcal{L}' = \{L'_\ell\} \cup \{(L \cup \{\ell\})/\!\!/p, \text{ for all } L \in \mathcal{L}\}$ and $\mathcal{I}' = \{(I \cup \{\ell\})/\!\!/p, \text{ for all } I \in \mathcal{I}\}$, then $\{U'\} \cup \mathcal{L}' \cup \mathcal{I}'$ is a partition of A'.

To prove that A' satisfies $(*)$, there remains to prove that these sets are mutually-tu. Suppose by contradiction that a non-tu set $X' \subset A'$ intersects several sets in $\{U'\} \cup \mathcal{L}' \cup \mathcal{I}'$, while X' contains no te-lace of $\mathcal{L}'$ and shares at most one row with each te-interlace of $\mathcal{I}'$. Let $X \subset A$ be such that $X' = X/\!\!/p$, with $\ell \in X$. Since ℓ is the only row index that differs between X' and X, the same holds for $X \setminus \{\ell\}$ with respect to $\mathcal{L}$ and $\mathcal{I}$, hence $X \setminus \{\ell\}$ is a tu-set by $(*)$ in A. If $X = A$, then necessarily $X = A = U \cup L_\ell$. Otherwise, by the induction hypothesis, X is a te-set. Since X is not a tu-set as trimming preserves tu-sets, it contains a te-lace by Lemma 3.7.9. This te-lace contains ℓ since $X \setminus \{\ell\}$ is a tu-set. If $\ell \in U$, ℓ belongs to no te-lace by the mutually-tu property, hence $\ell \notin U$. Therefore, $\ell \in L_\ell$ and $L_\ell \subseteq X$. But then, $L'_\ell \subseteq X'$, which contradicts the choice of X'. □

Since $A/\!\!/(\ell, j)$ has full row rank, it is equimodular by the above claim and the induction hypothesis. Note that this holds for all pivot (ℓ, j) of A, that is, for all $j \in \text{supp}(\ell)$. Therefore, Lemma 3.7.2 implies that A is equimodular.

Case 2. Every row of A is in a te-interlace.

In this case, by $(*)$, A is the disjoint union of te-interlaces $I_1, \ldots, I_t$ so that every set intersecting each I_i in at most one row is a tu-set. We may assume that $t \geqslant 2$, as otherwise we are done since a te-interlace is a te-set by definition.

Let ℓ be a row of I_1. Let $p = (\ell, j)$ be a pivot of A and let I'_i be the rescaled matrix obtained from I_i by p-trimming A, for $i = 1, \ldots, t$ and let $A' = \widehat{A/\!\!/p}$. By Claim 3.7.32, $I'_2, \ldots, I'_t$ are maximal te-interlaces of A'. Moreover, $I'_2, \ldots, I'_t$ satisfy $(*)$. Indeed, their union is equal to $(A \setminus (I_1 \setminus \{\ell\}))/\!\!/p$, and $A \setminus (I_1 \setminus \{\ell\})$ is a te-set by $|I_1| \geqslant 2$, the induction hypothesis, and Claim 3.7.30.

Since I_1 is a te-interlace, by Corollary 3.7.17, $I'_1 = \widehat{I_1/\!\!/p}$ is either a tu-set or a te-lace.

Claim 3.7.34. *The sets $I'_1, I'_2, \ldots, I'_t$ are mutually-tu.*

Proof. By contradiction, suppose that a minimal non-tu set X' intersects several of these sets, with $I'_1 \nsubseteq X'$ when I'_1 is a te-lace, and with $|X' \cap I'_i| \leqslant 1$, for $i \geqslant 2$, and let $X' = \widehat{X/\!\!/p}$. Since trimming preserves tu-sets, X is not a tu-set. Since $|X' \cap I'_i| \leqslant 1$ for all $i \geqslant 2$, we have $|X \cap I_i| \leqslant 1$ for all $i \geqslant 2$. By Claim 3.7.30, $A \setminus I_1$ satisfies $(*)$, hence it is a te-set by the induction hypothesis. Therefore $X \setminus I_1$ is a te-set, and hence in fact a tu-set, as otherwise it would contain a te-lace by Lemma 3.7.9 and contradict $(*)$ in $A \setminus I_1$.

Since I'_1 is either a te-lace or a tu-set, it satisfies $(*)$, hence $X' \setminus I'_1 \neq \varnothing$. Moreover, $X' \cap I'_1 \neq \varnothing$ because $I'_2, \ldots, I'_t$ satisfy $(*)$. In particular, $X' \cap I'_1$ and $X' \setminus I'_1$ are both tu-sets.

To sum up, X is the disjoint union of a te-interlace $I_X = X \cap I_1$ and a tu-set $U_X = X \setminus I_1$, and moreover $\widehat{I_X/\!\!/p}$ is a tu-set. The latter implies that $\text{eqdet}(I_X) = 2^{|I_X|-1}$. By Claim 3.7.32 and the fact that trimming preserves tu-sets, successively (r, j')-trimming X with respects to rows r of U_X maintains $(*)$, and hence yields $\text{eqdet}(X) = \text{eqdet}(I_X) = 2^{|I_X|-1}$.

If A is a square submatrix of X and A' the corresponding one in X', the definition of X gives the following, since the scaled rows to get X' are precisely the rows of $I_X \setminus \{\ell\}$: $\det(A) = 2^{|I_X|-1}\det(A')$. Thus $\det(A') = 0,\pm 1$ for all square submatrices A' of X'. Since X' is a minimal nontu-set, all its proper subsets are tu-sets. But then, X' is also a tu-set, and this contradiction finishes the proof. $\square$

Claim 3.7.34 implies that $A /\!\!/ p$ satisfies $(*)$, and hence is equimodular by the induction hypothesis. This holds for all $p = (\ell, j)$ with $j \in \mathrm{supp}(\ell)$, therefore A is equimodular by Lemma 3.7.2. $\square$

3.8 Proofs of the results of Section 3.6

3.8.1 Proofs of the results of Section 3.6.1: Hilbert bases

Recall that all the cones of interest are pointed.

From Theorem 3.2.3, we have the following.

Remark 3.8.1. Notice that if A is unimodular or a tu-set, the number of integer points in $\mathcal{Z}^<(A)$ is 1, and when A is a te-set, we have $\mathrm{gcddet}(A_I) = \mathrm{eqdet}(A_I)$, for every $I \subseteq \{1, \dots, k\}$.

Motivated by the definition of mutually-tu sets, we now introduce a more general concept. Any two disjoint sets of integer vectors A and B are *lattice orthogonal* when $A \sqcup B$ is linearly independent and $\mathrm{gcddet}(A \sqcup B) = \mathrm{gcddet}(A)\,\mathrm{gcddet}(B)$. In that case, there is a nice relation between the integer points of the associated parallelepipeds.

Corollary 3.8.2. *Let A and B be lattice orthogonal sets of vectors of $\mathbb{Z}^n$, then we have*

$$|\mathcal{Z}^<(A \sqcup B) \cap \mathbb{Z}^n| = |\mathcal{Z}^<(A) \cap \mathbb{Z}^n| \cdot |\mathcal{Z}^<(B) \cap \mathbb{Z}^n|.$$

Proof. By applying three times Theorem 3.2.3, and by the lattice orthogonality, we have:

$$\begin{aligned} |\mathcal{Z}^<(A \sqcup B) \cap \mathbb{Z}^n| &= \mathrm{gcddet}(A \sqcup B) \\ &= \mathrm{gcddet}(A)\,\mathrm{gcddet}(B) \\ &= |\mathcal{Z}^<(A) \cap \mathbb{Z}^n| \cdot |\mathcal{Z}^<(B) \cap \mathbb{Z}^n|, \end{aligned}$$

as desired. $\square$

Lemma 3.8.3. *The Hilbert basis of the Minkowski sum of cones generated by lattice orthogonal sets is the union of the Hilbert basis of each.*

Proof. Let $C = \mathrm{cone}(A_1 \sqcup \dots \sqcup A_k) = \mathrm{cone}(A_1) + \dots + \mathrm{cone}(A_k) = C_1 + \dots + C_k$ be the Minkowski sum of cones generated by lattice orthogonal sets $A_1, \dots, A_k$ of $\mathbb{R}^n$. Since $A_1 \sqcup \dots \sqcup A_k$ is linearly independent, each C_i is a face of C. By Lemma 3.2.1, we have $\bigcup_i \mathcal{H}(C_i) \subseteq \mathcal{H}(C)$. An immediate induction relying on Corollary 3.8.2 yields $|\mathcal{Z}^<(\bigcup_i A_i) \cap \mathbb{Z}^n| = \prod_i |\mathcal{Z}^<(A_i) \cap \mathbb{Z}^n|$. Therefore, any integer point in $\mathcal{Z}^<(\bigcup_i A_i)$ can be expressed as a non-negative integer combination of integer points in each $\mathcal{Z}^<(A_i)$. This yields $\mathcal{H}(C) \subseteq \bigcup_i \mathcal{H}(C_i)$, which concludes. $\square$

Corollary 3.5.8 implies the following, which also proves Lemma 3.6.2.

Remark 3.8.4. Mututally-tu te-bricks are lattice orthogonal.

In addition to Lemma 3.2.4, we will also use the following well-known formula in the proof of Theorem 3.6.3.

Lemma 3.8.5. *For every positive integer m, we have the following:*

$$\sum_{i \text{ even}}^{m} \binom{m}{i} = \sum_{i \text{ odd}}^{m} \binom{m}{i} = 2^{m-1}.$$

Proof. This comes from that both quantities equal the half sum of

$$\sum_{i \text{ even}}^{m} \binom{m}{i} + \sum_{i \text{ odd}}^{m} \binom{m}{i} = (1+1)^m = 2^m,$$

and

$$\sum_{i \text{ even}}^{m} \binom{m}{i} - \sum_{i \text{ odd}}^{m} \binom{m}{i} = \sum_{i \text{ even}}^{m} \binom{m}{i}(-1)^i + \sum_{i \text{ odd}}^{m} \binom{m}{i}(-1)^i = (1-1)^m = 0.$$

□

We can now give the proof of Theorem 3.6.3, which we restate below.

Theorem 3.8.6 (Theorem 3.6.3)**.** *Let $A = \{a^1, \dots, a^m\}$ be a te-brick and $C = \operatorname{cone}(A)$.*

1. *If A is a tu-set, then $\mathcal{H}(C) = A$.*
2. *If A is a te-lace, then $\mathcal{H}(C) = A \cup \{\frac{1}{2}\sum_j a^j\}$.*
3. *If A is a thin te-interlace, then $\mathcal{H}(C) = A \cup \left\{\frac{1}{2}(a^i + a^j)\right\}_{1 \leqslant i < j \leqslant m}$.*
4. *If A is a thick te-interlace, then one of the following holds:*
   a. $\mathcal{H}(C) = A \cup \left\{\frac{1}{2}(a^i + a^j)\right\}_{1 \leqslant i < j \leqslant m} \cup \left\{\frac{1}{4}\sum_j a^j\right\}$,
   b. $\mathcal{H}(C) = A \cup \left\{\frac{1}{2}(a^i + a^j)\right\}_{1 \leqslant i < j \leqslant m} \cup \left\{\frac{3}{4}a^i + \frac{1}{4}\sum_{j \neq i} a^j\right\}_{i \in \{1,\dots,m\}}$.

   Moreover, the number of 1's and -1's in each column of A have the same parity $p \in \{0,1\}$, and 4.a occurs if and only if $m \equiv 2p \pmod 4$.

Proof. Let $A = \{a^1, \dots, a^m\}$ be a te-brick and $C = \operatorname{cone}(A)$.

1. tu-sets

Simplicial cones generated by tu-sets have no nontrivial Hilbert basis elements.

2. te-laces

The only nontrivial Hilbert basis element of a simplicial cone generated by a te-lace is the half sum of its generators.

Now, suppose A is a te-interlace of size m.

Claim 3.8.7. $\left\{\frac{1}{2}(a^i + a^j)\right\}_{1 \leqslant i \leqslant j \leqslant m} \subseteq \mathcal{H}(C)$.

Proof. For $1 \leqslant i < j \leqslant m$, note that $\frac{1}{2}(a^i + a^j)$ is an integer point of C since the a^i's are ± 1 vectors. Moreover, $\operatorname{cone}\{a^i, a^j\}$ is a face of C, and $\mathcal{H}\left(\operatorname{cone}\{a^i, a^j\}\right) = \{a^i, a^j, \frac{1}{2}(a^i + a^j)\}$ since it is a cone generated by the te-lace $\{a_i, a_j\}$. By Lemma 3.2.1, each $\frac{1}{2}(a^i + a^j)$ is a Hilbert basis elements of C. □

3. Thin te-interlaces.

In this case, A has equideterminant 2^{m-1}. By Lemma 3.2.3, the number of points in $\mathcal{Z}^<(A)$ is 2^{m-1}. The result follows from Claim 3.8.7, Lemma 3.2.4, and the following claim.

Claim 3.8.8. *The number of distinct integer points of $\mathcal{Z}^<(A)$ which are non-negative integer combinations of $\left\{\frac{1}{2}(a^i+a^j)\right\}_{1\leqslant i\leqslant j\leqslant m}$ is 2^{m-1}.*

Proof. Let us denote the set of integer points of $\mathcal{Z}^<(A)$ which are non-negative combinations of $\left\{\frac{1}{2}(a^i+a^j)\right\}_{1\leqslant i\leqslant j\leqslant m}$ by X. Since the cone is simplicial, a point $x\in X$ is uniquely expressed as

$$x=\sum_{k=1}^{m}\lambda_k a^k.$$

Since x is in $\mathcal{Z}^<(A)$, we have $\lambda_k\in\{0,\frac{1}{2}\}$, for $k\in\{1,\dots,m\}$. Moreover, since x is integer and the a^k's are ±1, an even number of λ_k's are equal to $\frac{1}{2}$. Thus, the points of X are obtained by setting to $\frac{1}{2}$ an even number i of the λ_k's, and by setting the rest to 0 the $m-i$ remaining λ_k's. Each choice yields a different point since the a^k's are linearly independent. Therefore, by Lemma 3.8.5, we have

$$|X|=\sum_{i\text{ even}}\binom{m}{i}=2^{m-1}.$$

□

4. Thick te-interlaces.

By Corollary 3.7.18, we may assume that A is square and invertible. In this case, A has determinant 2^m. By Theorem 3.2.3, the number of points in $\mathcal{Z}^<(A)$ is 2^m. We first compute a candidate set for the Hilbert basis of cone(A). We have $\left\{\frac{1}{2}(a^i+a^j)\right\}_{1\leqslant i\leqslant j\leqslant m}\subseteq\mathcal{H}(C)$ by Claim 3.8.7. Moreover, the following holds.

Claim 3.8.9. *Either $\frac{1}{4}\sum_j a^j$ or $\left\{\frac{3}{4}a^i+\frac{1}{4}\sum_{j\neq i}a^j\right\}_{1\leqslant i\leqslant m}$ is contained in $\mathcal{Z}^<(A)\cap\ \mathbb{Z}^n$.*

Proof. By definition of the parallelepiped, all these points are in $\mathcal{Z}^<(A)$. By Lemma 3.7.12 and Corollary 3.7.19, we have:

$$A=\operatorname{diag}(\varepsilon)\begin{bmatrix}1 & \mathbf{1}^\top\\ \mathbf{1} & \mathbf{J}-2B\end{bmatrix}\operatorname{diag}(\mu),\tag{3.8.1}$$

for two signing matrices $\operatorname{diag}(\varepsilon)$ and $\operatorname{diag}(\mu)$ and a complement minimally non-totally unimodular 0,1 matrix B, of odd size by Theorem 3.7.10. The numbers of 1's and -1's have the same parity in every column of A, because $\operatorname{diag}(\varepsilon)A\operatorname{diag}(\mu)$ contains an even number of 1's and -1's in each column. In the first column, this is due to B being of odd size. In the other columns, it holds because B is 0,1 and minimally non-totally unimodular, and hence contains an even number of 1's in each column, by Theorem 3.7.7. Multiplying by the signing matrices then impacts the parity of the numbers of 1's and -1's similarly in every column of A.

There are two cases, according to whether this parity is (E) even or (O) odd.

(E) Suppose $m\equiv 0\pmod 4$. Then, since A has an even number of 1's and -1's in each column, the coordinates of $\sum_j a^j$ are all congruent to $0\pmod 4$. Therefore, $\frac{1}{4}\sum_j a^j$ is an integer point of C and is in $\mathcal{Z}^<(A)$.

Suppose $m\equiv 2\pmod 4$. Then, for $i\in\{1,\dots,m\}$, the coordinates of $3a^i+\sum_{j\neq i}a^j$ are all congruent to $0\pmod 4$. Therefore, $\frac{3}{4}a^i+\frac{1}{4}\sum_{j\neq i}a^j$ is an integer point of C and is in $\mathcal{Z}^<(A)$, for $i\in\{1,\dots,m\}$.

(O) This case is similar to the previous one, except that the situations $m \equiv 0 \pmod 4$ and $m \equiv 2 \pmod 4$ are reversed. □

□

According to Claim 3.8.9, there are two cases, namely cases 4.a and 4.b. In each case, we will apply Lemma 3.2.4 to determine the Hilbert basis. By Theorem 3.2.3, the parallelepiped spanned by A contains 2^m integer points.

4.a Let $S = \left\{\frac{1}{2}(a^i + a^j)\right\}_{1 \leqslant i \leqslant j \leqslant m} \cup \left\{\frac{1}{4}\sum_j a^j\right\}$.

Claim 3.8.10. *The number of distinct integer points of $\mathcal{Z}^<(A)$ which are non-negative integer combinations of S is 2^m.*

Proof. Let X denote the set of integer points of $\mathcal{Z}^<(A)$ obtained by non-negative integer combinations of points in S. Let $h = \frac{1}{4}\sum_j a^j$. A point $x \in X$ is uniquely expressed as

$$x = \sum_{k=1}^{m} \lambda_k a^k.$$

Since x is in $\mathcal{Z}^<(A)$, we have $\lambda_k \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$, for $k \in \{1, \ldots, m\}$. The points in X with all λ_k's in $\{0, \frac{1}{2}\}$ do not imply h in the combination, hence their number is 2^{m-1} as in the proof of Claim 3.8.8.

We need to compute the number of points of X having coefficients also in $\{\frac{1}{4}, \frac{3}{4}\}$. Since the a^k's are ± 1, such points are expressed as $h + \tilde{x}$, where $\tilde{x} = \sum_{k=1}^{m} \mu_k a^k$ is a point in X with all μ_k's in $\{0, \frac{1}{2}\}$, an even number of them being positive. We find all such $\tilde{x}$ by setting an even number of μ_k's to $\frac{1}{2}$ and the other ones to 0.

Thus, such points x are obtained by setting an even number of λ_k's to $\frac{3}{4}$ and the others to $\frac{1}{4}$, and, by Lemma 3.8.5, their number is

$$\sum_{i \text{ even}} \binom{m}{i} = 2^{m-1}.$$

Finally, we have $|X| = 2^{m-1} + 2^{m-1} = 2^m$. □

4.b Let $S = \left\{\frac{1}{2}(a^i + a^j)\right\}_{1 \leqslant i \leqslant j \leqslant m} \cup \left\{\frac{3}{4}a^i + \frac{1}{4}\sum_{j \neq i} a^j\right\}_{1 \leqslant i \leqslant m}$.

Claim 3.8.11. *The number of integer points of $\mathcal{Z}^<(A)$ which are non-negative integer combinations of S is 2^m.*

Proof. Let us denote this set of integer points by X and let $h^i = \frac{3}{4}a^i + \frac{1}{4}\sum_{j \neq i} a^j$, for $i \in \{1, \ldots, m\}$. A point $x \in X$ is uniquely expressed as

$$x = \sum_{k=1}^{m} \lambda_k a^k.$$

Since x is in $\mathcal{Z}^<(A)$, we have $\lambda_k \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$, for $k \in \{1, \ldots, m\}$. Since we are considering points in the parallelepiped, the points in X with all λ_k's in $\{0, \frac{1}{2}\}$ imply no h^i's in the combination, hence their number is 2^{m-1} as in the proof of Claim 3.8.8.

We need to compute the number of points x of X having λ_k's also in $\{\frac{1}{4}, \frac{3}{4}\}$, whose decomposition involves at least one h^i. Since $x \in \mathcal{Z}^<(A)$, its decomposition involves at most one h^i, hence $h^i + \tilde{x}$, where $\tilde{x} = \sum_{k=1}^m \mu_k a^k$ is a point in X with all μ_k's in $\{0, \frac{1}{2}\}$, an even number of them being positive, and $\mu_i = 0$. Thus, every such x is obtained by setting to $\frac{3}{4}$ an odd number of λ_k's, one coming from h^i and the others from the nonzero μ_k's. The other λ_k's are $\frac{1}{4}$. Hence, by Lemma 3.8.5, their number is

$$\sum_{s \text{ odd}}^{m} \binom{m}{s} = 2^{m-1}.$$

Finally, this implies $|X| = 2^m$. □

In both cases 4.a and 4.b, note that no point of S is an integer combination of the others. The conclusion then comes from Lemma 3.2.4 together with either Claim 3.8.10 or Claim 3.8.11. □

3.8.2 Proofs of the results of Section 3.6.2: triangulations

We first prove that the join of the triangulations of each individual cone generated by a te-brick yields a regular unimodular Hilbert triangulation. This section ends with a proof of Corollary 3.1.2.

Recall that the join of two triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ of two cones generated by two disjoint sets of vectors whose union is linearly independent is the triangulation $\mathcal{T}_1 * \mathcal{T}_2 = \{C_1 + C_2 \colon C_1 \in \mathcal{T}_1, C_2 \in \mathcal{T}_2\}$.

Lemma 3.8.12 (Lemma 3.6.5)**.** *The join of the regular unimodular Hilbert triangulations of cones generated by disjoint mutually-tu te-bricks whose union is linearly independent is a regular unimodular Hilbert triangulation of their Minkowski sum.*

Proof. By Remark 3.8.4, the te-bricks are lattice orthogonal. Hence, by Lemma 3.8.12, the join of the triangulations of each individual cone generated by disjoint mututally-tu te-bricks is a Hilbert triangulation. Let $C = \mathrm{cone}(A_1 \sqcup \cdots \sqcup A_k) = \mathrm{cone}(A_1) + \cdots + \mathrm{cone}(A_k) = C_1 + \cdots + C_k$ be the Minkowski sum of simplicial cones generated by mutually-tu te-bricks $A_1, \ldots, A_k$, with $A = A_1 \sqcup \cdots \sqcup A_k$ linearly independent. Let $C'_1 + \cdots + C'_k$ be a cone in the join, where $C'_i = \mathrm{cone}(B_i)$ is a unimodular cone in the triangulation of C_i and $B_i \subseteq \mathcal{H}(C_i)$, for $i = 1, \ldots, k$. Since A_i and B_i are sets of linearly independent vectors and span the same subspace, we have $B_i = Q_i A_i$, for some square invertible matrices Q_i. In particular, we have $\mathrm{gcddet}(B_i) = \mathrm{eqdet}(B_i) = |\det(C'_i)| = 1$. Consequently, $B = QA$, where $B = B_1 \sqcup \cdots \sqcup B_k$ and Q is the block diagonal matrix whose blocks are the Q_i's. Combining the previous remarks and the fact that $A_1, \ldots, A_k$ are lattice orthogonal, we compute the determinant of the simplicial cone $C'_1 + \cdots + C'_k$, whose generators are $B_1 \sqcup \cdots \sqcup B_k$:

$$\begin{aligned}
\det(C'_1 + \cdots + C'_k) &= \pm \,\mathrm{gcddet}(B_1 \sqcup \cdots \sqcup B_k) \\
&= \pm \left(\prod_i \det(Q_i)\right) \mathrm{gcddet}(A_1 \sqcup \cdots \sqcup A_k) \\
&= \pm \left(\prod_i \det(Q_i)\right) \left(\prod_i \mathrm{gcddet}(A_i)\right) \\
&= \pm \prod_i \det(Q_i)\, \mathrm{eqdet}(A_i) \\
&= \pm \prod_i \mathrm{eqdet}(B_i) \\
&= \pm 1
\end{aligned}$$

Therefore, the join is also a unimodular triangulation. The Minkowski sum of two simplicial cones generated by two sets of vectors whose union is linearly independent is a simplicial cone as it combinatorially corresponds to the join of two affinely independent simplices, which is again a simplex. Moreover, the join preserves regularity by [87, Section 2.3.2]. □

Thanks to Theorem 3.8.6 and Lemma 3.8.12 all that remains to find is a regular unimodular Hilbert triangulation of the cones generated by each type of te-brick, to finally obtain the following.

Theorem 3.8.13 (Theorem 3.6.6)**.** *Let A be a te-set without thick te-interlace of size greater than six. Then,* $\operatorname{cone}(A)$ *has a regular unimodular Hilbert triangulation.*

Proof. Let $A = \{a^1, \dots, a^m\}$ be a te-brick and $C = \operatorname{cone}(A)$.

1. A is a tu-set.

The regular unimodular Hilbert triangulation is the cone itself.

2. A is a te-lace.

The stellar triangulation at $h = \frac{1}{2}\sum_j a^j$, namely the one formed by the n cones generated by h and $m-1$ generators among m, is regular since its coincides with the strong pulling at h which preserved regularity [87, Lemma 2.1]. For $j \in \{1, \dots, m\}$, we have $\operatorname{gcddet}(\{h\} \cup \{a^i : i \neq j\}) = \frac{1}{2}\operatorname{eqdet}(A) = 1$, which yields the unimodularity. Finally, all the cones are generated by Hilbert basis elements.

3. A is a thin te-interlace.

Inspired by the regular triangulation in [61], we start this case with some definitions. Let $G = (V, E(G))$ be a graph. An edge of G whose ends coincide is called a *loop*. Let $\{e^v\}_{v \in V}$ be the canonical basis of $\mathbb{R}^V$. If $ij \in E(G)$ is an edge of G, its *characteristic vector* is $\chi^{ij} = e^i + e^j$. The characteristic vector of a loop ii is $\chi^{ii} = e^i + e^i = 2e^i$. The *incidence matrix* $A_G \in \{0,1\}^{|E(G)| \times |V|}$ is the matrix whose rows are the characteristic vectors of the edges of G. A *spanning* subgraph of G is a connected graph $H = (V, F)$ with $F \subseteq E(G)$.

Let $\mathring{K}_m$ denote a complete graph with m vertices to which we added a loop ii at each vertex i. Embed $\mathring{K}_m$ as a convex m-gon in $\mathbb{R}^2$, with clockwise labeled vertices $v_1, \dots, v_m$, edges ij embedded as line segments $[v_i, v_j]$, for each $i \neq j$, and loops ii as circles outside the m-gon meeting the m-gon only at v_i. The edges of $\mathring{K}_m$ encode the Hilbert basis elements of C as follows: an edge ij represents $\frac{1}{2}(a^i + a^j)$ and a loop ii represents a^i. We say that two distinct edges *meet* if the associated segments of the convex m-gon labeled by these numbers intersect. This happens either if they have a common end, or if the edges are ik and jl with $i < j < k < l$. A loop *meets* an edge if they share a vertex. A *stellar cycle* of this embedding $\mathring{K}_m$ is a spanning subgraph with m pairwise meeting edges or loops. Let $\mathcal{S}_m$ denote the set of stellar cycles of $\mathring{K}_m$. As a loop is considered as a cycle of length 1, note that a stellar cycle contains a unique cycle which is odd. Consequently, for each loop, there is precisely one stellar cycle whose unique cycle is this loop.

Remark 3.8.14. We mention that stellar cycles are in one-to-one correspondence with odd sets of vertices of $\mathring{K}_m$. Indeed, given an odd number of vertices of $\mathring{K}_m$, there is a unique cycle on these vertices whose edges pairwise intersect. Then, all the remaining edges are uniquely determined since they have to meet all the edges of the cycle. Therefore, up to permutation of rows or columns, the incidence matrix A_S of a stellar cycle S is a square matrix of size $m \times m$ of the form:

$$\begin{bmatrix} A_D & \mathbf{0} \\ * & I_{m-|D|} \end{bmatrix},$$

for D the odd cycle of S ($A_D = \begin{bmatrix} 2 \end{bmatrix}$ if the cycle is a loop). Since D is an odd cycle, its incidence matrix A_D has determinant ± 2, therefore $\det(A_S) = \pm 2$. We refer the reader to [61] for more details.

Claim 3.8.15. *The collection $\mathcal{T}$ of the cones $C_S = \operatorname{cone}(\frac{1}{2}(a^i + a^j) \colon ij \in E(S))$, for all $S \in \mathcal{S}_m$, forms a regular unimodular Hilbert triangulation of C.*

Proof. By Theorem 3.8.6, all the cones in $\mathcal{T}$ are generated by Hilbert basis elements. Note that a triangulation of $P = \operatorname{conv}(A)$ yields a triangulation of C. Let $\Delta_{2,A} = \operatorname{conv}\left(\frac{1}{2}(a^i + a^j) \colon i \neq j\right)$ and $T_i = \operatorname{conv}\left(a^i, \frac{1}{2}(a^i + a^j) \colon j \neq i\right)$, for $i \in \{1, \dots, m\}$. All these sets have disjoint relative interiors and we have $P = \Delta_{2,A} \cup \bigcup_{i=1}^m T_i$. The simplex T_i shares only the facet $F_i = \operatorname{conv}(\frac{1}{2}(a^i + a^j) \colon j \neq i)$ with $\Delta_{2,A}$, for $i \in \{1, \dots, m\}$. Therefore, from any triangulation of $\Delta_{2,A}$, we obtain one for P by attaching the simplex T_i at each facet F_i.

By [61, Theorem 2.3 and Lemma 2.4] the set of simplices

$$\left\{ \operatorname{conv}\left(\frac{1}{2}(e^i + e^j) \colon ij \in E(S)\right), \text{ for all loopless } S \in \mathcal{S}_m \right\}$$

forms a regular triangulation of $\Delta_2 = \operatorname{conv}\left(\frac{1}{2}(e^i + e^j) \colon i \neq j\right)$.
Therefore, the simplices $\operatorname{conv}\left(\frac{1}{2}(a^i + a^j) : ij \in E(S)\right)$ for all loopless $S \in \mathcal{S}_m$, form a regular triangulation of $\Delta_{2,A}$, as $\Delta_{2,A} = A^\top \Delta_2$ is the image of Δ_2 by the linear map $A^\top$.

We now attach the T_i's. The triangulation we obtain remains regular by assigning a sufficiently large weight to every a^i, which is separated from the other a^j's by the facet F_i, and those facets have maximal weight among the other ones of the triangulation. Since the cones associated with these simplices are the cones C_S defined above for all loopless S, the collection of Hilbert cones $\mathcal{T}$ is indeed a regular triangulation of C.

All that remains to show is that each cone of $\mathcal{T}$ is unimodular. Recall that A is thin and $\operatorname{eqdet}(A) = \pm 2^{m-1}$. For $i \in \{1, \dots, m\}$, the cone C_i associated with the stellar cycle with loop ii is generated by a_i and $\frac{1}{2}(a^i + a^j)$ for $j \neq i$. It is unimodular since:

$$\det(C_i) = \pm \operatorname{eqdet}\left(\frac{1}{2}\begin{bmatrix} \mathbf{I}_{i-1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0}^\top & 2 & \mathbf{0}^\top \\ \mathbf{0} & \mathbf{1} & \mathbf{I}_{m-i} \end{bmatrix} A\right) = \pm \frac{1}{2^m} 2 \operatorname{eqdet}(A) = \pm 1.$$

For a stellar cycle $S \in \mathcal{S}_m$, by Remark 3.8.14, we have:

$$\det(C_S) = \pm \operatorname{eqdet}\left(\frac{1}{2} A_S^\top A\right) = \pm \frac{1}{2^m} \det(A_S) \operatorname{eqdet}(A) = \pm 1,$$

as desired. $\square$

4. A is a thick te-interlace of size at most 6.

There are four cases: either m equals 4 or 6, and either 4.a or 4.b occurs in Theorem 3.8.6. In each case, we used Polymake [76] to generate a regular Hilbert triangulation, inspired by [61], and a simple determinant computation algorithm to check unimodularity. In Section 3.8.3, the reader will find figures, generated from the output of our Polymake script, representing the cones of the triangulations for each of these cases. Each figure corresponds to a set of m Hilbert basis elements generating a unimodular Hilbert cone in one of the four regular unimodular Hilbert triangulations. More precisely, in each figure there are n vertices labeled from 1 to m and colored loops and edges, which correspond to Hilbert basis elements as follows.

In all figures:

- a^i is represented by a blue loop at vertex i, for $i = 1, \dots, m$,
- $\frac{1}{2}(a^i + a^j)$ is represented by a blue edge ij, for $i \neq j$.

In Case 4.a, the additional Hilbert basis element is:

- $\frac{1}{4} \sum_j a^j$, which is represented by a green dot in the center of the figure.

In Case 4.b, there are m additional Hilbert basis elements, which are:

- $\frac{3}{4} a^i + \frac{1}{4} \sum_{j \neq i} a^j$, which are each represented by a red circle around vertex i, for $i = 1, \dots, m$.

All the associated figures are in Section 3.8.3. □

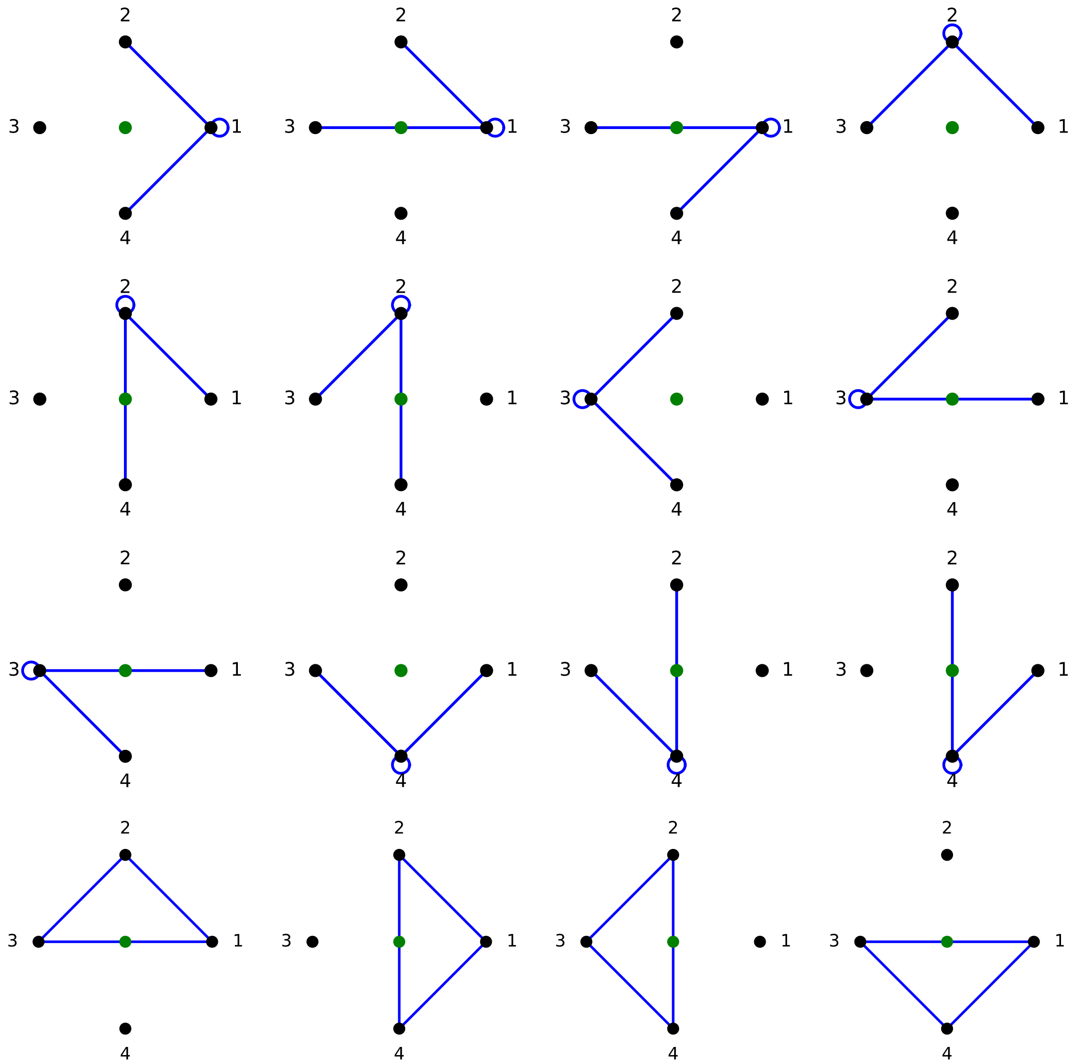
Finally, we prove Corollary 3.1.2.

Corollary 3.8.16 (Corollary 3.1.2)**.** *Simplicial box-totally dual integral cones in the non-negative orthant have the integer Carathéodory property.*

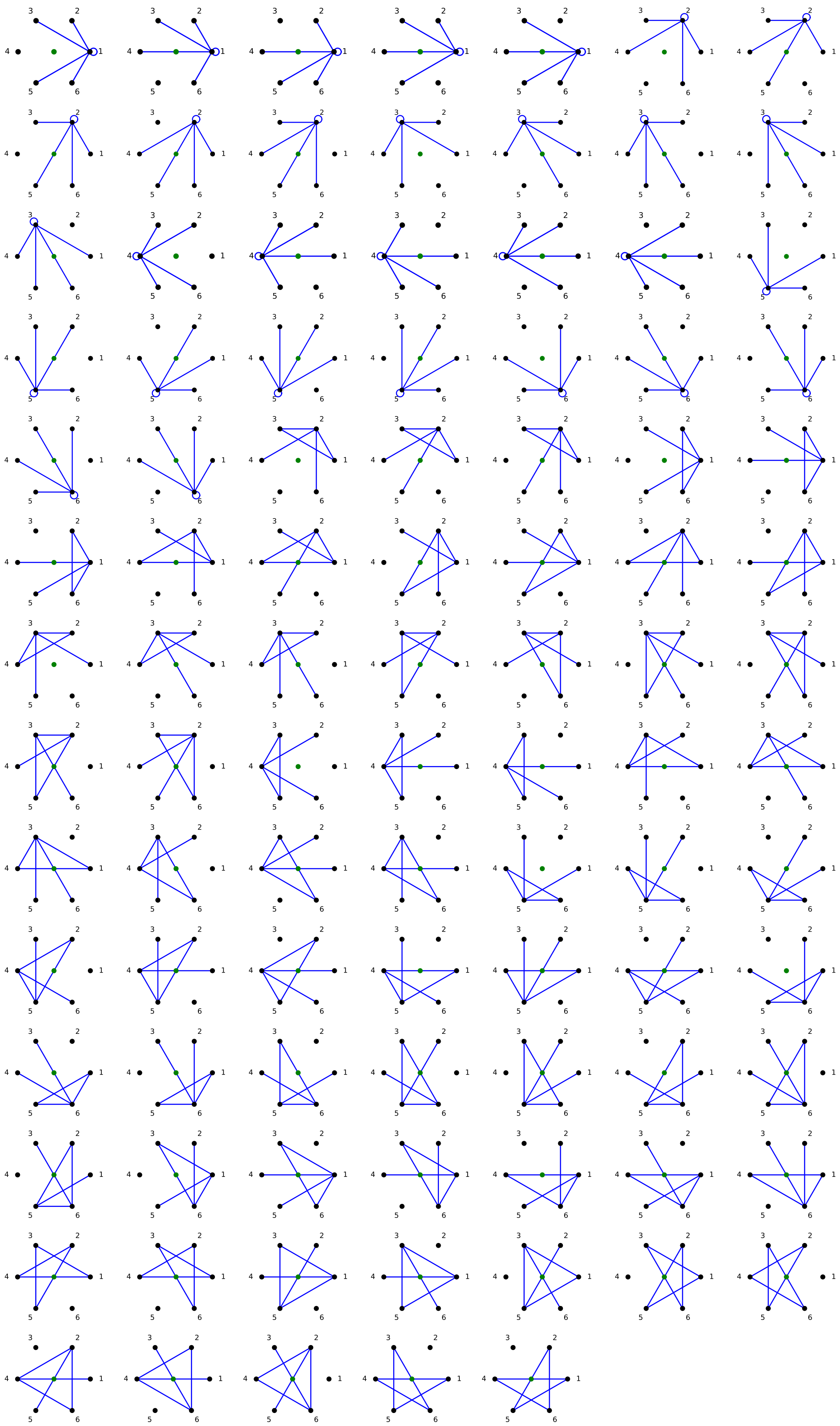
Proof. As mentioned in the proof of Theorem 3.5.5, a simplicial cone is box-TDI if and only if it is generated by a te-set. If the cone is in the non-negative orthant, then the te-set is 0,1. Since 0,1 te-sets contain no te-interlaces, Theorem 3.8.13 provides a regular unimodular Hilbert triangulation of the cone, hence it has the integer Carathéodory property. □

3.8.3 Figures for the case 4. of Theorem 3.6.3

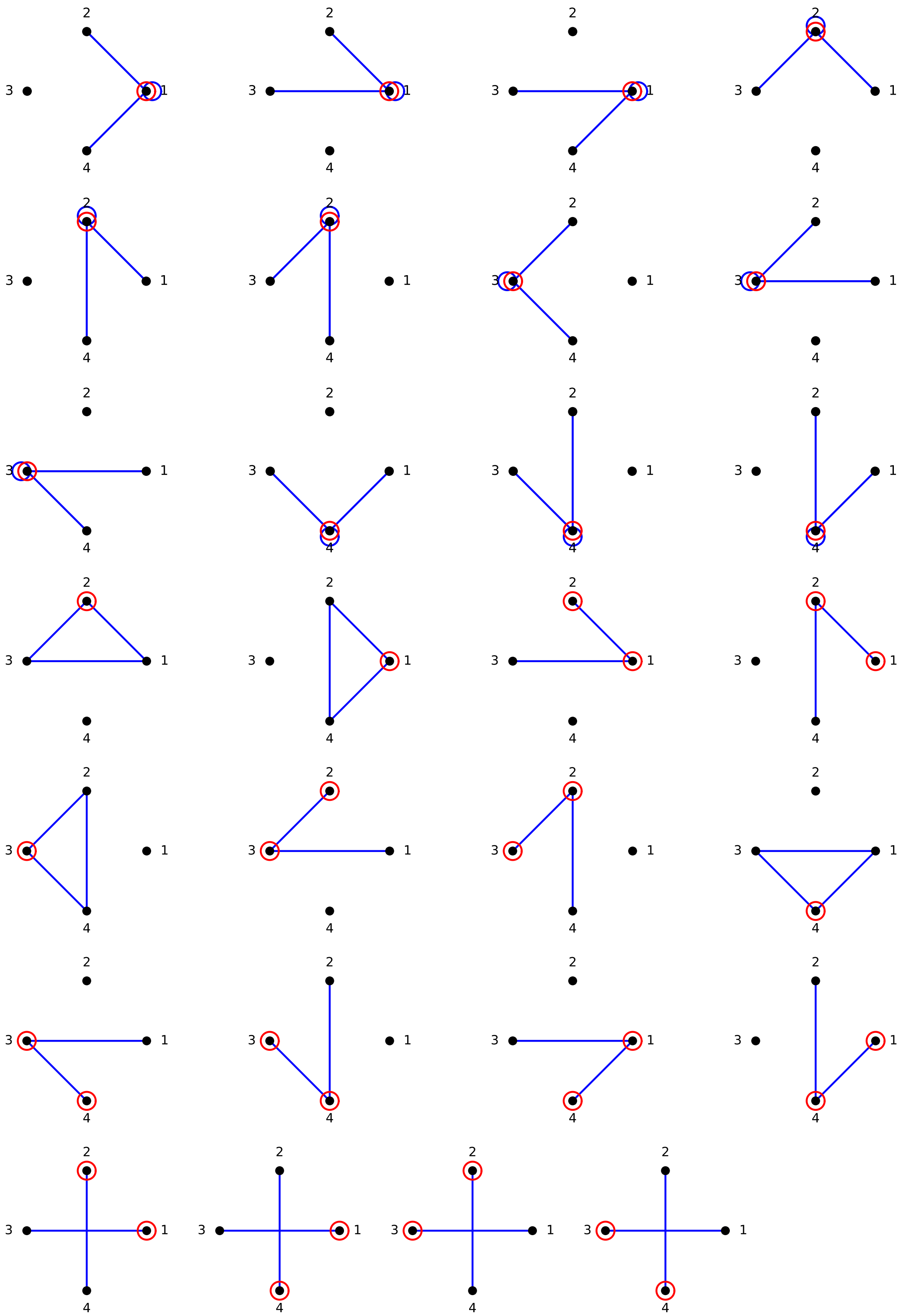
$m = 4$, Case 4.a:



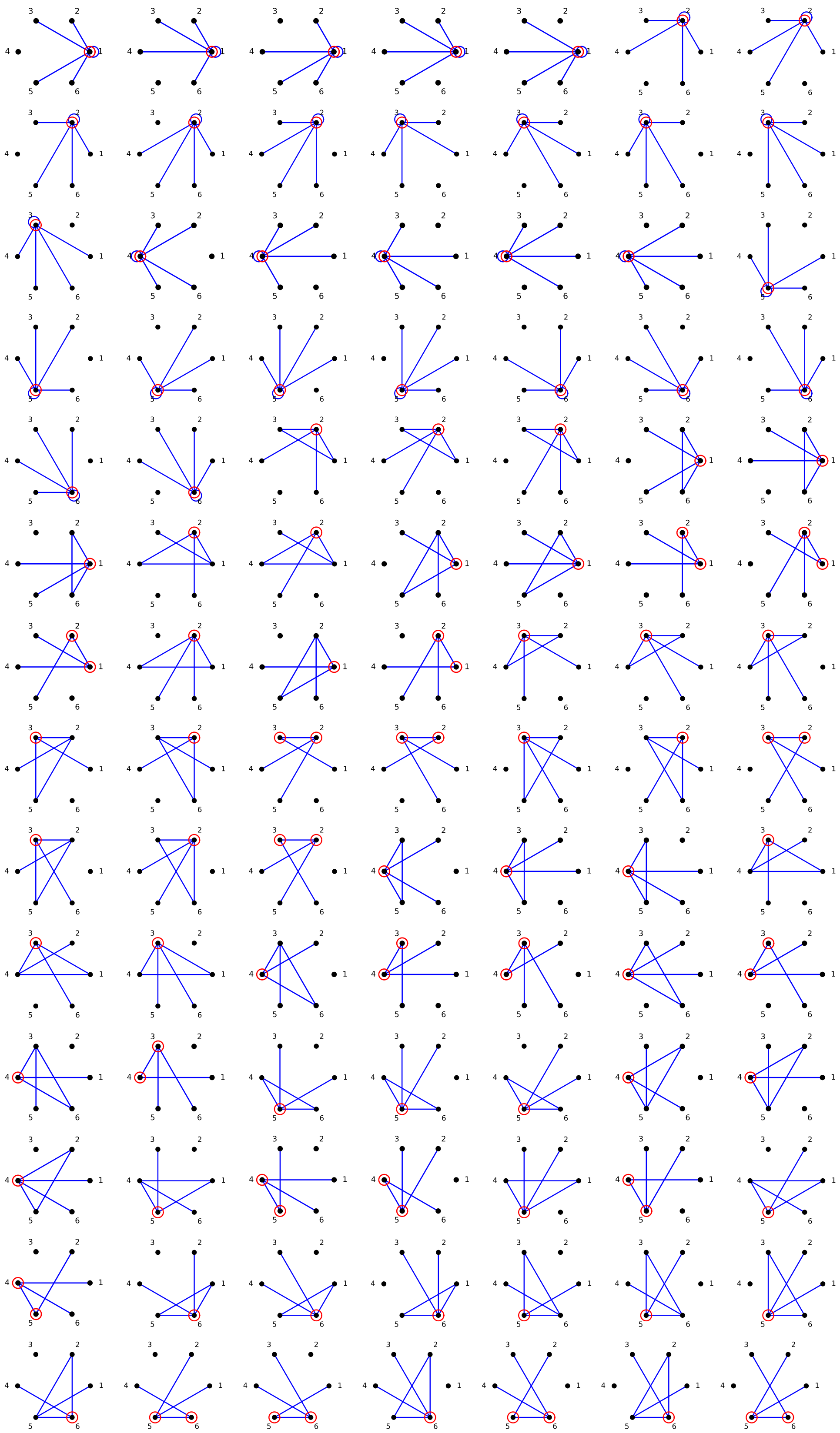
$m = 6$, Case 4.a:

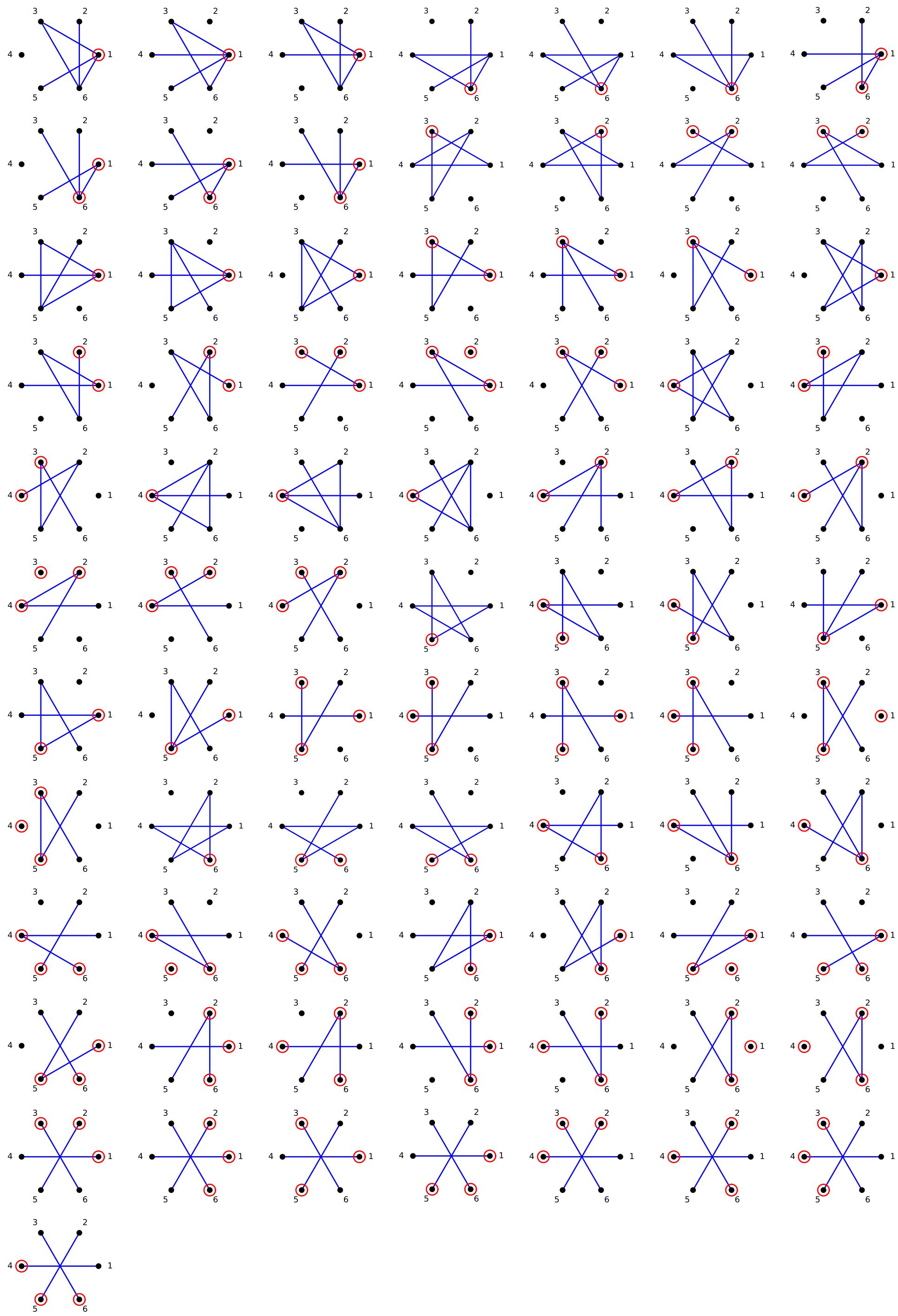


$m = 4$, Case 4.b:



$m = 6$, Case 4.b:





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4 Toric-colorable seeds with Picard number 4

This chapter is devoted to Frank H. Lutz.

This chapter is based on a joint work with Suyoung Choi and Hyeontae Jang published in Crelle's journal [39] [1]. We solve the first step of the characterization of toric manifolds with Picard number 4 by finding every toric-colorable seeds with Picard number 4.

4.1 Introduction

Our interest lies at the intersection of geometry, with the classification of non-singular complete toric varieties, and discrete mathematics, with the enumeration of piecewise linear (PL) spheres.

Toric geometry

A *toric variety* of complex dimension n is a normal algebraic variety over the field of complex numbers $\mathbb{C}$ that admits an effective algebraic action of $(\mathbb{C}^*)^n$ having a dense orbit. The *fundamental theorem for toric geometry* states that the classification of toric varieties of complex dimension n is equivalent to that of *fans* in $\mathbb{R}^n$. The *cone* generated by a finite set of rational vectors $R \subset \mathbb{R}^n$ is $\operatorname{cone}(R) = \{\sum_{r\in R} \alpha_r r \colon \alpha_r \geqslant 0\}$. A *fan* in $\mathbb{R}^n$ is a collection of cones that is closed under taking faces and such that the intersection of any pair of them is a face of both. The one-dimensional cones of a fan are called the *rays*. The combinatorial structure of a fan Σ in $\mathbb{R}^n$ having m rays is represented by a pair (K, λ) where:

— K is the underlying complex of Σ, with vertex set $[m] = \{1, \dots, m\}$, whose face lattice is isomorphic to that of Σ, and

— $\lambda\colon [m] \to \mathbb{Z}^n$ is a map that is one-to-one assigning vertices of K to the primitive generator of the rays of Σ.

The fan is therefore given by $\Sigma = \{\operatorname{cone}(\{\lambda(i)\}_{i\in\sigma}) : \sigma \in K\}$. In this chapter, we are interested in compact smooth toric varieties, simply *toric manifolds*, that are characterized by *complete non-singular* fans, whose pairs (K, λ) satisfy:

— K is a simplicial complex, which is a PL sphere, and

— λ satisfies the *non-singularity condition* for K: for any face $\{i_1, \dots, i_k\}$ in K, the set $\{\lambda(i_1), \dots, \lambda(i_k)\}$ is *non-singular*, namely it is part of a basis of $\mathbb{Z}^n$.

1. A preliminary version was published in a conference proceedings [38].

For a simplicial complex K on $[m]$ with $\dim(K) = n-1$, its *Picard number* is $\mathrm{Pic}(K) = m - n$. When K is obtained from a complete non-singular fan, this number is the *Picard number* of the associated toric manifold, see [73, Section 3.4]. Kleinschmidt [99] and Batyrev [17] classified toric manifolds of “small” Picard numbers 2 and 3, respectively.

The classification of complete non-singular fans necessitates the classification of PL spheres that can serve as their underlying simplicial complexes. For Picard number $\leqslant 3$, every PL sphere is polytopal by Mani [105], and can therefore be described using Gale diagrams [82]. Using this, one can characterize which PL spheres support a complete non-singular fan [80, 67].

In this chapter, we take one more step and focus on the case of Picard number 4. However, the same method is hardly applicable for Picard number 4 since 3-dimensional Gale diagrams are difficult to use. Moreover, a non-polytopal PL sphere of Picard number 4 exists as shown in [83], and there also exists a complete non-singular fan whose underling simplicial complex can be non-polytopal [131]. Therefore, we approach the problem using other combinatorial properties of PL spheres, such as their property of being *weak pseudo-manifolds*.

Enumeration of weak pseudo-manifolds and PL spheres

The enumeration of triangulations of manifolds has been a longstanding challenge since the end of the 19th century. The advances made in this area have provided valuable tools for researchers studying discrete and PL geometry. Computer-assisted enumeration has been a major approach for tackling these problems and we follow this direction in the present chapter.

In particular, the aforementioned works focus on the enumeration of weak pseudo-manifolds, pseudo-manifolds, PL manifolds, PL spheres, and polytopal spheres. Let K be a simplicial complex on $[m]$. It is *pure* if its maximal faces are all of the same size. These top dimensional faces are called the *facets* of K, and are of size $\dim(K) + 1$. The faces of size $\dim(K)$ are called the *ridges*. A simplicial complex is a *weak pseudo-manifold* if it is pure and every ridge is contained in exactly two facets. Additionally, it is a *pseudo-manifold* (without boundaries) if its ridge-facet graph is connected. One example of a pseudo-manifold is the boundary of the $(n-1)$-simplex $\partial\Delta^{n-1}$ whose facets are the subsets of size n of $[n+1]$, and has Picard number 1. Any set of affinely independent points $v_1, \ldots, v_{n+1} \in \mathbb{R}^n$ yields a geometric realization $|\partial\Delta^{n-1}|$ of $\partial\Delta^{n-1}$ that is homeomorphic to the sphere S^{n-1}. A simplicial complex K of dimension $n-1$ is a PL sphere if there exists a subdivision of K and a subdivision of $\partial\Delta^{n-1}$ such that these subdivisions are isomorphic. It is a *PL manifold* if the link of each of its faces is a PL sphere. A *polytopal sphere* is the boundary complex of a simplicial polytope. We have the following hierarchy on simplicial complexes:

$$\begin{array}{c}\text{polytopal}\\\text{spheres}\end{array} \subseteq \text{PL spheres} \subseteq \text{PL manifolds} \subseteq \begin{array}{c}\text{pseudo-}\\\text{manifolds}\end{array} \subseteq \begin{array}{c}\text{weak}\\\text{pseudo-}\\\text{manifolds}\end{array} \subseteq \begin{array}{c}\text{pure}\\\text{simplicial}\\\text{complexes.}\end{array}$$

The notable advance in enumerating these simplicial complexes progresses along two directions: small dimensions and small Picard numbers. It is well-known that PL spheres of dimension 2 are equivalent to 3-connected planar graphs, and they can be generated using the `plantri` algorithm, as demonstrated in [20], for up to 23 vertices. Enumerations of polytopal or non-polytopal PL spheres of dimension 3 with 8 and 9 vertices are provided in [6, 7], following the work in [83], and [5], respectively. While the enumeration of PL spheres of Picard number $\leqslant 3$ has been accomplished in [82], the enumeration for Picard number 4 remains an open problem.

The enumerations of all weak pseudo-manifolds of dimension 2 with 7 and 8 vertices are documented in [58] and [59], respectively. Lutz and Sulanke extensively used a new method based on lexicographic enumeration to obtain (weak) pseudo-manifolds of dimensions 2 and 3, with up to 12 and 11 vertices, as detailed in [104, 130]. Additionally, in [11], a characterization of pseudo-manifolds of Picard number $\leqslant 3$ is provided.

We challenge the enumeration of PL spheres and weak pseudo-manifolds of Picard number 4. In this chapter, we introduce a new method that consists in representing a pure simplicial complex as a $\{0,1\}$-vector, allowing for the use of linear algebra for fast computations, and adaptability to graphic processing unit (GPU) programming. In Section 4.2, readers can find an explicit algorithm described in the Compute Unified Device Architecture (CUDA) language for enumerating weak pseudo-manifolds whose facets are in a given input set and satisfy affine conditions on the associated vector.

(Real) Buchstaber number and classification problems in toric topology

Without any assumption on the simplicial complex K on $[m]$, we construct a topological space called the *polyhedral product* $(\underline{X},\underline{Y})^K$ of K with respect to a pair (X,Y) of topological spaces which is

$$(\underline{X},\underline{Y})^K \coloneqq \bigcup_{\sigma\in K} \{(x_1,\dots,x_m)\in X^m \mid x_i\in Y \text{ when } i\notin\sigma\}\,.$$

The *moment-angle complex* $\mathcal{Z}_K$ of K is $(\underline{D^2},\underline{S^1})^K$, and the *real moment-angle complex* $\mathbb{R}\mathcal{Z}_K$ of K is $(\underline{D^1},\underline{S^0})^K$, where D^d represents the d-dimensional disk, and S^{d-1} denotes its boundary sphere of dimension $d-1$. We observe that the T^1-action on the pair (D^2,S^1) yields the canonical action of the m-dimensional torus $T^m=(S^1)^m$ on $\mathcal{Z}_K$. The *Buchstaber number* $s(K)$ is the maximal integer r for which there exists a subtorus of rank r acting freely on $\mathcal{Z}_K$.

Similarly, there is an S^0-action on the pair (D^1,S^0). For clarity and consistency in our terminology throughout this chapter, we will treat S^0 as the additive group $\mathbb{Z}_2=\mathbb{Z}_2\mathbb{Z}$ with two elements $\{0,1\}$. This yields the canonical $\mathbb{Z}_2^m$-action on $\mathbb{R}\mathcal{Z}_K$. The *real Buchstaber number* $s_{\mathbb{R}}(K)$ is defined by the maximal rank r of a subgroup of $\mathbb{Z}_2^m$ acting freely on $\mathbb{R}\mathcal{Z}_K$. Determination of the Buchstaber and real Buchstaber numbers is one of the central questions in toric topology, and it has been actively studied in many literatures such as in [72, 68, 88, 9, 16].

It is noteworthy that when an r-dimensional subtorus H of T^m acts freely on $\mathcal{Z}_K$, the resulting quotient space $\mathcal{Z}_K/H$ supports a well-behaved torus action $T^m/H\cong T^{m-r}$ with an orbit space that exhibits a reverse face structure isomorphic to K. It is known that $s(K)\leqslant s_{\mathbb{R}}(K)\leqslant \operatorname{Pic}(K)$, see [68]. In particular, a toric manifold associated with K is topologically obtainable from the quotient of $\mathcal{Z}_K$ by a free action of subtorus of dimension $m-n$, see [60, 26] for details. Consequently, considering PL spheres whose Buchstaber number is maximal, that is equal to $m-n$, is of significant importance.

A simplicial complex of dimension $n-1$ is said to be *toric colorable* if it has a maximal Buchstaber number, and $\mathbb{Z}_2^n$*-colorable* if it has a maximal real Buchstaber number. We have the following hierarchy on PL spheres:

$$\text{fan-like} \quad\subseteq\quad \text{toric colorable} \;\subseteq\; \mathbb{Z}_2^n\text{-colorable},$$

where a *fan-like* PL sphere is a simplicial complex that is the underlying complex of some complete non-singular fan. Therefore, the first step for classifying toric manifolds of Picard number 4 is to characterize toric colorable PL spheres of Picard number 4. This derives the following classification problem.

Problem 4.1.1. Find which PL spheres of Picard number 4 have (real) Buchstaber number 4.

The key tool for obtaining the answer lies in the finiteness of the problem. This result stems from Choi and Park [46], who established that there exists a finite set of PL spheres, called toric colorable *seeds*, from which all toric colorable PL spheres can be derived through iterated *wedge operations* (this operation is also referred to as the *J-construction* in [14]). The maximal number of vertices of a toric colorable seed of Picard number $p \geqslant 3$ is $2^p - 1$. Consequently, we only need to enumerate the toric colorable seeds up to $n = 11$.

In this chapter, we use that toric colorable seeds must belong to certain binary matroids, as detailed in Section 4.3. This restricts the number of facets inputted into our GPU algorithm, allowing us to obtain results for n up to 10. In addition, we mathematically address the extreme case $n = 11$ to further reduce the algorithmic complexity and derive the following main theorem.

Theorem 4.1.2. *Up to isomorphism, the number of toric (or $\mathbb{Z}_2^n$-)colorable seeds of dimension $n-1$ and Picard number $p \leqslant 4$ is as follows:*

	n											
p	1	2	3	4	5	6	7	8	9	10	11	> 11	*total*
1	1												1
2		1											1
3		1	1	1									3
4		1	4	21	142	733	1190	776	243	39	4		3153

with the empty slots displaying zero.

The database containing the toric colorable seeds of Picard number 4, the CUDA script, and a C++ version of the script (which we used for both performance comparison and verification purposes) are available on my Github repository:

`https://github.com/MVallee1998/GPU_handle`

From [46], we obtain this corollary of Theorem 4.1.2 which completely solves Problem 4.1.1.

Corollary 4.1.3. *The toric (or $\mathbb{Z}_2^n$-)colorable PL spheres of dimension $n-1$ and Picard number $p \leqslant 4$ are precisely those obtained after consecutive wedge operations on the toric (or $\mathbb{Z}_2^n$-)colorable seeds (of Theorem 4.1.2).*

In summary, the set of toric (or $\mathbb{Z}_2^n$-)colorable PL spheres of dimension $n-1$ and Picard number $p \leqslant 4$ is "finitely generated" through multiple wedge operations on the explicit $1+1+3+3153$ seeds outlined in Theorem 4.1.2.

Application to the space of rational curves on toric manifolds

We anticipate that our theorem contributes to the understanding of toric manifolds of Picard number 4. For instance, in this chapter, we employ it to address a question posed by Chen, Fu, and Hwang [32] in 2014 for this specific case.

Let X be a toric manifold whose corresponding fan has m rays and $\mathrm{RatCurves}(X)$ the normalized space of rational curves on X. Fix an irreducible component $\mathcal{K}$ of $\mathrm{RatCurves}(X)$. Then we have a universal family $\rho\colon \mathcal{U} \longrightarrow \mathrm{RatCurves}(X)$, which is a complex projective line bundle, and $\mu\colon \mathcal{U} \longrightarrow X$, which is an evaluation map. The component $\mathcal{K}$ is called *minimal* if μ is dominant and $\mu^{-1}(x)$ is complete for a general point $x \in X$. The *degree* of $\mathcal{K}$ is defined as the degree of the intersection of the anti-canonical divisor of X with any member in $\mathcal{K}$. In [32], it is shown that the sum of the degrees of all minimal components is less than or equal to m, and it is asked when the equality holds. In Section 4.5 we answer this question for every toric manifold of Picard number 4.

4.2 Classification of weak pseudo-manifolds by graphic processing unit computing

In this section, we provide a general approach on how to use graphic processing unit (GPU) parallel computing capability for classifying weak pseudo-manifolds with given properties.

4.2.1 Enumerating weak pseudo-manifolds

Let K be a pure simplicial complex of dimension $n-1$ on the vertex set $[m] = \{1, 2, \dots, m\}$. A *facet* of K is a face of size n, and a *ridge* is a face of size $n-1$. We denote by $\mathcal{F}(K)$ and $\mathcal{R}(K)$ the sets of facets and ridges of K, respectively. We will often use the words facets and ridges without specifying a simplicial complex when referring to a subset of size n and a subset of size $n-1$ of $[m]$. Technically, they refer to the facets and ridges of the simplicial complex whose facets are the subsets of size n of $[m]$. We shall provide an algorithm as follows:

Input: A set $\mathcal{F}$ of facets, and a collection $\mathcal{G}$ of affine functions on $\mathbb{R}^{\mathcal{F}}$, called *properties*.
Output: The set of weak pseudo-manifolds K such that $\mathcal{F}(K) \subseteq \mathcal{F}$ and $g(\mathcal{F}(K)) > 0$ for all $g \in \mathcal{G}$, namely, K *satisfies* all the properties.

Provided any set of facets $\mathcal{F} = \{F_1, \dots, F_M\}$, we can compute the set $\mathcal{R} = \{r_1, \dots, r_N\}$ of all ridges that come from these facets. We then construct the *ridge-facet incidence matrix* $A(\mathcal{F}) = (a_{i,j})$ of size $N \times M$ as follows:

$$a_{i,j} = \begin{cases} 1 & r_i \subset F_j \\ 0 & \text{otherwise} \end{cases},$$

for $i = 1, \dots, N$ and $j = 1, \dots, M$. A simplicial complex K whose facets are all in some set of facets $\mathcal{F} = \{F_1, \dots, F_M\}$ can be regarded as a *characteristic vector* $K = (k_1, \dots, k_M)^t \in \mathbb{Z}^M$ with

$$k_j = \begin{cases} 1 & F_j \in K \\ 0 & F_j \notin K \end{cases},$$

for $j = 1, \dots, M$. The pure simplicial complex K is a *weak pseudo-manifold* if any ridge of K is in exactly two facets of K. That reflects in the following property:

Proposition 4.2.1. *Let $\mathcal{F}$ be a set of facets, $A = A(\mathcal{F})$ the ridge-facet incidence matrix of $\mathcal{F}$, and K a pure simplicial complex whose facets are all in $\mathcal{F}$. Then K is a weak pseudo-manifold if and only if the coordinates of the product AK are all in* $\{0, 2\}$.

From that, in $\mathbb{Z}_2^M$, the characteristic vectors of weak pseudo-manifolds are all included in the $\mathbb{Z}_2$-kernel of the matrix A, seen as a linear map $A\colon \mathbb{Z}_2^M \to \mathbb{Z}_2^N$.

Let $B = \begin{bmatrix} K_1 & \cdots & K_s \end{bmatrix}$ be a matrix whose columns form a $\mathbb{Z}_2$-basis of $\ker_{\mathbb{Z}_2} A$. Every weak pseudo-manifold K is uniquely expressed as one of the 2^s possible $\mathbb{Z}_2$-linear combinations of $K_1, \dots, K_s$, namely $K = \sum_{i=1}^{s} x_i K_i \pmod 2 = BX$, for $X = (x_1, \dots, x_s)^t \in \mathbb{Z}_2^s$. We find a suitable basis $\widetilde{K}_1, \dots, \widetilde{K}_s$ to reduce the number of cases to compute.

We first explain how to construct this basis when the set $\mathcal{F} = \binom{[m]}{n}$ contains all possible facets of $[m]$ and $\mathcal{R} = \binom{[m]}{n-1}$ all the ridges. There are $\binom{m}{n}$ facets and $\binom{m}{n-1}$ ridges. For a ridge r, we will write as $(AK)_r$ the coordinate of AK corresponding to r. Let us denote by $\mathcal{P}(r) := \{j \in [M]\colon r \subset F_j\}$ the set of the indexes in $\mathcal{F}$ of the facets containing r, called the *parents* of r, that are the only facets contributing to $(AK)_r$. In this first case, any ridge has $m-n+1$ parents. For a kernel matrix B whose row are indexed by $\mathcal{F}$, let us denote by $B_{\mathcal{P}(r)}$

the matrix whose rows are the ones of B taken at indexes $\mathcal{P}(r)$. For every $r \in \mathcal{R}$, for every $t = 1, \dots, s$, the tth column of $B_{\mathcal{P}(r)}$ has an even number of ones since the basis element K_t has an even number of facets containing r. Performing a mod 2 Gaussian elimination on the columns of $B_{\mathcal{P}(r)}$ yields a matrix of the following form

$$B_{\mathcal{P}(r)}E = \begin{bmatrix} Z_{m-n} & \mathbf{0} \end{bmatrix},$$

with the $(k+1) \times k$-matrix

$$Z_k = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ 1 & \cdots & 1 & 1 \end{bmatrix},$$

for some integer k, and $E \in \mathrm{GL}(s, \mathbb{Z}_2)$ corresponding to the operations performed in the Gaussian elimination. The columns of the new matrix BE correspond to a convenient basis of the $\mathbb{Z}_2$-kernel of A. Indeed, only its first $m-n$ columns have facets contributing to $(AK)_r$. Moreover, taking the mod 2 linear combination of strictly more than two of them makes $(AK)_r$ be strictly greater than 2, which is a case we want to avoid computing since we focus on weak pseudo-manifolds, see Proposition 4.2.1. Thus, this decreases the number of $\mathbb{Z}_2$-linear combinations containing the first $m-n$ new generators that we need to compute from 2^{m-n} to $\binom{m-n}{0} + \binom{m-n}{1} + \binom{m-n}{2}$.

By writing $r^1 := r$ and $E_1 := E$, one can inductively repeat the latter process by taking care at step $k+1$ of:

— choosing each time a new ridge r^{k+1} such that for all $i = 1, \dots, k, \mathcal{P}(r^i) \cap \mathcal{P}(r^{k+1}) = \varnothing$, and

— starting the Gaussian pivot at columns index $k(m-n)+1$ so that the structure of the generators of previous columns is not lost.

This process terminates at some step $k_{\max}$ whenever one of the former conditions cannot be satisfied. We obtain a final matrix, whose columns are the new basis elements $\widetilde{K}_1, \dots, \widetilde{K}_s$, and, up to reordering, whose rows are according to the sets $\mathcal{P}(r^1), \dots, \mathcal{P}(r^{k_{\max}})$ looks as follows:

$$BE_1 \cdots E_{k_{\max}} = \left[\begin{array}{ccccc} Z_{m-n} & 0 & \cdots & \cdots & 0 \\ 0 & Z_{m-n} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & Z_{m-n} & 0 \\ \hline \star & \star & \star & \star & \star \end{array}\right] = \begin{bmatrix} \widetilde{K}_1 & \cdots & \widetilde{K}_s \end{bmatrix}.$$

In this case, we decrease the total number of $\mathbb{Z}_2$-linear combinations from 2^s to $(1 + (m-n) + \binom{m-n}{2})^{k_{\max}} 2^{s - k_{\max}(m-n)}$ since we should take at most 2 basis elements for each block Z_{m-n} and there remains $s - k_{\max}(m-n)$ columns $\widetilde{K}_i$ after these blocks.

As for the general case, there may be ridges having less than $m-n+1$ parents. In this case, we try to wisely choose some ridges $r^1, \dots, r^{k_{\max}}$ such that the blocks Z_k are of the maximum possible size so we minimize the number of $\mathbb{Z}_2$-linear combinations BX of the generators we need to compute. That provides a partition $I_1, \dots, I_l$ of $\{1, \dots, s\}$ such that if we are to sum more than two basis elements with indexes in I_k, for $k = 1, \dots, l$, we are sure not to obtain a weak pseudo-manifold. We can split the vector X in the $\mathbb{Z}_2$-linear combinations BX as blocks according to this partition: $X = \sum_{k=1}^{l} X_k$, with X_k representing the part of X whose only nonzero coordinates are in I_k. Let us denote by $\mathcal{X}_k$ the set of all such possible X_k for $k = 1, \dots, l$.

If we recap our process, given a set of facets $\mathcal{F}$, we constructed

- the ridge-facet incidence matrix A whose $\mathbb{Z}_2$-kernel contains all weak pseudo-manifolds,
- a matrix B whose columns form a convenient basis $\widetilde{K}_1, \dots, \widetilde{K}_s$ of $\ker_{\mathbb{Z}_2}(A)$,
- a partition $I_1, \dots, I_l$ of $\{1, \dots, s\}$,
- the sets $\mathcal{X}_1, \dots, \mathcal{X}_l$ of partitions of the vectors of $\mathbb{Z}_2^s$ such that for all $k = 1, \dots, l$, $X_k \in \mathcal{X}_k$ has a maximum of two nonzero coordinates which are all in I_k,

such that any weak pseudo-manifold whose facets are in $\mathcal{F}$ is of the form $K = BX$, with $X = \sum_{k=1}^{l} X_k$ for some $(X_1, \dots, X_l) \in \mathcal{X}(\mathcal{F}) = \mathcal{X}_1 \times \cdots \times \mathcal{X}_l$, satisfying that the entries of AK corresponding to the chosen ridges are in $\{0, 2\}$. Moreover, given any affine function $K \mapsto g(K)$, it is easy to check using computer programming that $g(K) > 0$ is verified.

4.2.2 Generalities about GPU programming

In this chapter, we used *Nvidia Compute Unified Device Architecture* (CUDA) [107]. One will notice that the syntax and vocabulary may differ from other GPU languages.

The general idea behind GPU computing is that it allows parallelizing tasks with two layers of parallel programming without requiring a supercomputer. Parallel programming takes several forms, and the two we will use are the following.

- Data parallelism: one has a list of elements $\mathcal{X}$ and wants to apply the same function g to every element $X \in \mathcal{X}$. In this case, each call of the function g is independent.
- Task parallelism: one has an element X and wants to apply a set of similar functions $g_1, \dots, g_k$ on X in order to obtain the result as a list $(g_1(X), \dots, g_k(X))$. The simplest example is a matrix product AX, and if each row of A is denoted by a_i, then the functions g_i are the inner products with the a_is.

In all that follows, a *thread (of execution)* will be a processing unit that computes machine operations linearly, and a GPU will be a two-layered structure of threads. Namely, a GPU will be a set of p *grids*, and each grid will be a set of q threads. Therefore a GPU can be seen as $p \times q$ threads organized for parallel programming, as in Figure 4.1. The number $p \times q$ of GPU threads that can run simultaneously is roughly the number of CUDA cores (if we consider Nvidia GPUs) and is around seventeen thousand for the current architectures (as of 2024). Thus a single GPU would be approximately equivalent to at least a thousand central processing unit (CPU) threads.

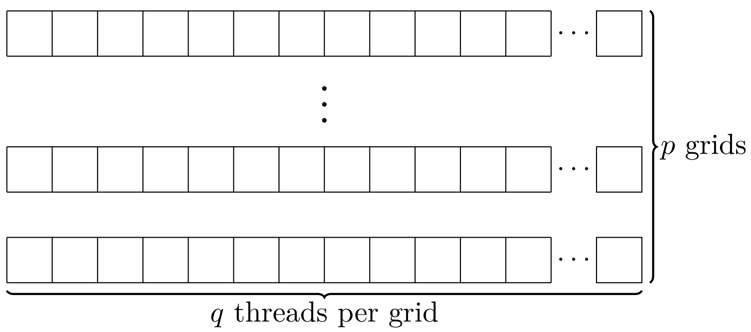

Figure 4.1 – The two layered parallel structure of a GPU.

In CUDA programming we use this two-layered structure as follows:

— **First layer (blocks):** Let $\mathcal{X} = \{X_1, \ldots, X_N\}$ be the set of data on which we want to apply the same function g, called the *kernel.* We create some list of N *blocks* indexed by an integer i. Each block embodies the function call $g(X_i)$. A block has three possible states: *on hold*, *active*, and *completed.* In the beginning, every block is on hold. Then the p grids of the GPU are filled with some blocks which will be running, these are active, and the rest are waiting to be launched on the grid and remain on hold. Whenever some active block has completed, the GPU replaces it with a block on hold. The program terminates when all blocks are completed.

— **Second layer (threads):** Whenever we send a block to a grid, the operations made in the block are split into threads using task parallelism, and we distribute any procedures in g into q functions which will run simultaneously on all q threads of the grid. Notice that we need every thread to finish its tasks to obtain the result. We can explicitly require this condition by *synchronizing* the threads.

In all that follows, we will use such notations:

— a set $\mathcal{X}$ is denoted as a list `list_X`,

— a matrix $A = [a_{i,j}]$ is represented as an array `A` whose entry at index i, j is `A[i][j]`$= a_{i,j}$, and

— a binary vector $X \in \mathbb{Z}_2^k$ is represented as a binary variable `x` on k bits.

We will use the following processor instructions on binary variables [107]:

— the *bitwise and* operation `x&y`, 64 operations per cycle,

— the *bitwise exclusive-or* operation `x^y`, 64 operations per cycle,

— the *population count* operation `popcount(x)` which counts the number of “1” bits, called *active* bits, in the value of `x`, 16-32 operations per cycle, and

— *atomic operations*, that we use to avoid memory access errors when many threads may want to write at the same memory location concurrently. The processor scheduler creates a queue of all atomic operation calls.

A *cycle* is the shortest time interval considered in a processor unit that it performs at its frequency. If the frequency is 1GHz the processor realizes 10^9 cycles per second.

The thread synchronization allows us to manage how the threads behave in parallel as follows:

— The `syncthreads()` command asks all the threads to wait for each others and come across the same line in the algorithm code of the kernel.

— For a local thread variable `test`, the `syncthreads_and(test)` and `syncthreads_or(test)` command allows us to manage the *and* and the *or* operation over all of the variables `test` existing in each thread of a grid. For example, if a thread encounters a condition that should stop the current case in a loop, then all the threads should stop at once since it is useless to compute this case.

4.2.3 The GPU algorithm for classifying weak pseudo-manifolds

To simplify our explanations, we suppose that there are $s = 64$ generators and that we can write the product $\mathcal{X}_1 \times \cdots \times \mathcal{X}_l$ as $\mathcal{X}_a \times \mathcal{X}_b$ such that $\mathcal{X}_a$ and $\mathcal{X}_b$ describe the 32 first and last generators, respectively. We thus decompose K as $K_a + K_b$, with $K_a = BX_a$ and $K_b = BX_b$ for every $(X_a, X_b) \in \mathcal{X}_a \times \mathcal{X}_b$. Both vectors X_a and X_b are binary vectors whose nonzero coordinates are in the 32 first or last coordinates, respectively, which we store as 32 bits variables `xa` and `xb`, namely as unsigned integers.

The dot product in $\mathbb{Z}$ of the binary forms `x` and `y` of two integers x and y, respectively, is the number `popcount(x&y)` of active bits of the *bitwise and* operation. Its mod 2 reduction is `popcount(x&y)&1`, the value of its least significant bit.

The main idea of the algorithm is to use M threads to compute each coordinate of $K \in \mathbb{Z}_2^M$, with M being the number of facets in $\mathcal{F}$, as provided in Algorithm 4.2.2.

Algorithm 4.2.2. The GPU algorithm for enumerating weak pseudo-manifolds.

Input: The list `list_F`, corresponding to the set of facets $\mathcal{F}$, and the list `list_G`, corresponding to the set of affine functions $\mathcal{G}$.

Output: The list `list_K` of weak pseudo-manifolds `K` with facets in `list_F` and that satisfy `g(K)>0` for every `g` in `list_G`.

Procedure INITIALIZATION
    Compute the ridge-facet incidence matrix $A = A(\mathcal{F}) \in \mathbb{Z}_2^{N\times M}$ and store it in `A`, a column sparse matrix: `A[k][i]` represents the index of the `k`th nonzero coordinate of the `i`th column of A.
    Compute $B = \begin{bmatrix} \widetilde{K_1} & \cdots & \widetilde{K_{64}} \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ \vdots & \vdots \\ a_M & b_M \end{bmatrix}$ and store it as two lists `list_a` and `list_b` of integers, where `list_a[k]` and `list_b[k]` represents the binary value of the row vectors a_k and b_k, respectively.
    Enumerate $\mathcal{X}_a$ and $\mathcal{X}_b$, and store them as two lists `list_Xa` and `list_Xb`.
    Create a list `list_Ka` of all the K_as:
    for all `xa` $\in$ `list_Xa` **do**
        for `k` $= 1, \ldots, M$ **do**
            `Ka[k]`$\leftarrow$`popcount(a[k]&xa)&1`
        end for
    end for
end Procedure
Shared memory: Integer array `r` of size N, such that `r[k]` stores the `k`th coefficient of the product AK.
Function KERNEL(`xa`,`Ka`)
    `i`$\leftarrow$local thread index
    `b`$\leftarrow$`list_b[i]`
    `ka`$\leftarrow$`list_Ka[i]`
    for all `xb` $\in$ `list_Xb` **do**
        `skip`$\leftarrow$`False`
        `Ki`$\leftarrow$`(popcount(b&xb)^ka)&1`
        `syncthreads()`
        for all `g` $\in$ `list_G` **do**
            compute `g`(K) using the thread values `Ki`
            if `g`$(K) \leqslant 0$ **then**
                `skip`$\leftarrow$`True`
                break
            end if
        end for
        if `syncthreads_or(skip)` **then**
            continue to the next `xb`
        end if
        Reinitialize each value of `r` to `0` using the threads
        if `Ki`$=$`1` **then**

```
            for k = 1, ..., n do
                increment r[A[k][i]] using the atomic add operation
                if r[A[k][i]] ⩾ 3 then
                    skip←True
                    break
                end if
            end for
        end if
        if syncthreads_or(skip) then
            continue to the next xb
        end if
        Add K to the list of results list_K
    end for
end Function
Procedure Main
    Launch the |X_a| blocks that correspond to all the pairs (xa,Ka) on the Kernel.
end Procedure
```

Remark 4.2.3. When we say "using the threads", we mean we evenly distribute the operations to perform among the threads. For example, to reinitialize the array `r`, we use the fact that we have q threads that can set to zero q coordinates simultaneously until all coordinates reset. Thus, it requires $\lceil \frac{N}{q} \rceil$ iterations, where N is the number of ridges. We use a similar process for calculating the image by the affine functions $g \in \mathcal{G}$.

Remark 4.2.4. We use the atomic add operation for incrementing values in `r` since many threads may write at the same memory location `r[k]`.

The global complexity of this algorithm is

$$\mathcal{O}\left(\frac{|\mathcal{X}_a|}{p} \times |\mathcal{X}_b| \times \frac{N}{q} \times (\alpha|\mathcal{G}| + 1)\right),$$

where α is the average complexity of the atomic operation when called multiple times for a given $g \in \mathcal{G}$.

4.3 Preparation for applying the algorithm

4.3.1 Finiteness of the problem and seedness

A simplicial complex K of dimension $n-1$ is a *PL sphere* if it has a subdivision that is isomorphic to any of $\partial\Delta^{n-1}$. It is a *PL manifold* if the link of each of its faces is a PL sphere. It is known that a PL sphere is a PL manifold, see [93, Lemma 1.17]. We refer the reader to [11] for more detailed definitions about PL manifolds.

A PL sphere K is called a *seed* if it is not a wedge of another PL sphere L. The following proposition follows immediately from the definition of the wedge.

Proposition 4.3.1 (seedness from minimal non-faces)**.** *A PL sphere K is a seed if and only if it satisfies the seedness condition; there is no face $\{v, w\}$ in K such that for every minimal non-face σ of K, we have either $\{v, w\} \cap \sigma = \{v, w\}$ or $\varnothing$.*

Remark that the seedness condition can be defined for general simplicial complexes.

Since the links of both new vertices in $\mathrm{wed}_v(K)$ are isomorphic to K, if $\mathrm{wed}_v(K)$ is a PL sphere, so is K. The converse also holds.

Proposition 4.3.2. *Let K be a PL sphere and v a vertex of K. Then* $\mathrm{wed}_v(K)$ *is a PL sphere.*

Proof. Suppose that K is an $(n-1)$-dimensional PL sphere. Since the suspension $K*\partial\{w_1,w_2\}$ of K is isomorphic to an edge subdivision of $\mathrm{wed}_v(K)$, both have the same PL structure. Moreover, $K*w_1$ is a subdivision of an n-simplex since K is a PL sphere. Hence $K*\partial\{w_1,w_2\} = (K*w_1)\cup_K (K*w_2)$ is a subdivision of the boundary of an n-simplex. □

Let K be a simplicial complex on the vertex set $[m]$. A *characteristic map* over K is a map $\lambda\colon [m] \longrightarrow \mathbb{Z}^n$ satisfying the non-singularity condition for K: for each face σ of K, $\lambda(\sigma)$ is a non-singular set. Then, it is known that the existence of characteristic maps over K is equivalent to the maximality of the Buchstaber number of K, i.e., $s(K) = m-n$, see [26]. We call K *toric colorable* if it admits a characteristic map.

One can also consider its mod 2 analogue. A mod 2 characteristic map over K is a map $\lambda^{\mathbb{R}}\colon [m] \longrightarrow \mathbb{Z}_2^n$ satisfying that $\lambda^{\mathbb{R}}(\sigma)$ is a linearly independent set for all $\sigma \in K$. Similarly, we call K *$\mathbb{Z}_2^n$-colorable* if it admits a mod 2 characteristic map.

Proposition 4.3.3 ([69], [44])**.** *Let K be an $(n-1)$-dimensional PL sphere and v a vertex of K. Then K is toric colorable if and only if so is* $\mathrm{wed}_v(K)$*, and K is $\mathbb{Z}_2^n$-colorable if and only if* $\mathrm{wed}_v(K)$ *is $\mathbb{Z}_2^{n+1}$-colorable.*

Notice that the composition of a characteristic map over K and mod 2 reduction $\mathbb{Z}^n \to \mathbb{Z}_2^n$ yields a mod 2 characteristic map over K. As a consequence, we firstly focus on $\mathbb{Z}_2^n$-colorable seeds.

We often see a mod 2 characteristic map $\lambda^{\mathbb{R}}$ as a *characteristic matrix*

$$\begin{bmatrix}\lambda^{\mathbb{R}}(1) & \lambda^{\mathbb{R}}(2) & \cdots & \lambda^{\mathbb{R}}(m)\end{bmatrix}.$$

Up to isomorphism, we may assume that the facet $\{1,2,\dots,n\}$ is in K. With this assumption, to check $\mathbb{Z}_2^n$-colorability, it is enough to consider mod 2 characteristic maps of the form $\lambda^{\mathbb{R}} = \begin{bmatrix} I_n & M\end{bmatrix}$ since the non-singularity for K is preserved by the left multiplication with an element of $\mathrm{GL}(n,\mathbb{Z}_2)$.

Let us define *dual characteristic maps (DCM)* over K. For $\lambda^{\mathbb{R}} = \begin{bmatrix} I_n & M\end{bmatrix}$, the DCM associated with $\lambda^{\mathbb{R}}$ is a map $\overline{\lambda^{\mathbb{R}}}\colon [m] \longrightarrow \mathbb{Z}_2^{m-n}$ such that

$$\overline{\lambda^{\mathbb{R}}} = \begin{bmatrix}\overline{\lambda^{\mathbb{R}}}(1) & \overline{\lambda^{\mathbb{R}}}(2) & \cdots & \overline{\lambda^{\mathbb{R}}}(m)\end{bmatrix}^t = \begin{bmatrix} M \\ I_{m-n}\end{bmatrix}.$$

We shorten the term *injective DCM* to *IDCM.*

Theorem 4.3.4 ([46])**.** *Let K be an $(n-1)$-dimensional PL sphere with m vertices and v, w distinct vertices of K. Then the following statements are true.*

1. *If every facet of K contains v or w, then K is a wedge or a suspension with respect to v and w.*
2. *If K is a seed that is not a suspension, then every DCM over K must be an IDCM.*

Statements (2) and (3) both imply:

3. *If K is a seed and $m-n \geqslant 3$, then $m \leqslant 2^{m-n}-1$.*

We conclude from Statement (3) of Theorem 4.3.4 that there are only finitely many $\mathbb{Z}_2^n$-colorable seeds of fixed Picard number p. We now focus on the case $p=4$. By Statement (3) of Theorem 4.3.4, we have $n \leqslant 11$, implying that it is enough to enumerate colorable seeds of dimension up to 10 ($n = 11$).

4.3.2 Checking isomorphism using minimal non-faces

One demanding problem when enumerating simplicial complexes is dealing with isomorphism. If a simplicial complex K has m vertices, then there are $m!$ possible relabeling for K. Given two simplicial complexes K and L, with respective vertex sets V and W, we want to find if they are isomorphic. One solution is to use McKay's graph isomorphism algorithm [106] on the face posets of K and L. We provide here a different solution for testing isomorphism by using their sets of minimal non-faces (MNF).

For every vertex v of K, we define its *color sequence* $c_K(v)$ by the increasing sizes of the minimal non-faces of K containing v. For example, if the set of minimal non-faces of K is $\{\{1,2,3\},\{3,4\},\{4,5,6\},\{2,6\},\{1,6\}\}$, then $c_K(1) = (2,3)$, $c_K(5) = (3)$, and $c(6) = (2,2,3)$. The color sequence of a vertex is preserved under isomorphism. The procedure for checking the existence of an isomorphism between two simplicial complexes K and L is the following.

1. Check whether K and L have the same combinatorial aspects such as the numbers of faces, and the numbers of non-faces.
2. Check whether $\{c_K(v)\colon v \in [m]\} = \{c_L(v)\colon v \in [m]\}$, by counting repetitions, using their MNF sets.
3. We give partitions $V_1,\dots,V_k$ and $W_1,\dots,W_k$ of V and W with respect to color sequences in K and L, respectively. We compute every relabeling $\phi_i\colon V_i \to W_i$ for every $i = 1,\dots,k$. They provide every relabeling $\phi = \phi_1\times\cdots\times\phi_k\colon V \to W$ that preserves the color sequences. If one ϕ sends one-to-one the minimal non-faces of K to the ones of L then K is isomorphic to L.

The number of relabeling that we compute is $(|V_1|!)\times\cdots\times(|V_k|!)$ instead of $|V|!$. This provides a fine improvement when there are many different color sequences and only a few vertices share the same color sequence.

4.3.3 Collecting PL spheres among weak pseudo-manifolds

We need a criterion for a weak pseudo-manifold to be a PL sphere. We obtain this criterion in two steps. First, when the Picard number is small enough, there is a nice characterization of PL manifolds that are PL spheres.

Theorem 4.3.5 ([12])**.** *Let K be a PL manifold such that $\mathrm{Pic}(K) \leqslant 7$. If K is a $\mathbb{Z}_2$-homology sphere, then it is a PL sphere.*

By using the above theorem and by the definition of PL manifolds, we obtain the following lemma.

Lemma 4.3.6 (PL sphereness)**.** *A weak pseudo-manifold K of Picard number $\leqslant 7$ is a PL sphere if and only if the link of any face (including the empty face) of K is a $\mathbb{Z}_2$-homology sphere.*

Proof. The "only if" part is immediate, so it is enough to show the "if" part. Suppose that the link of any face of K is a $\mathbb{Z}_2$-homology sphere. By applying Theorem 4.3.5, let us prove that the link of each face of K is a PL sphere.

We use induction on the dimension of the link of a face. We remark that the link of each $(n-2)$-face of K is the 0-sphere S^0 by the definition of weak pseudo-manifolds. In particular, it is a PL sphere. For $k \leqslant n-3$, let σ be a k-face of K, and $L = \mathrm{lk}_K(\sigma)$ its link. Note that $\mathrm{lk}_L(v)$ for a vertex v of L is equal to $\mathrm{lk}_K(\{v\}\cup\sigma)$. Therefore, if the link of any $(k+1)$-face of K is a PL sphere, then L is a PL manifold. By assumption, L is a $\mathbb{Z}_2$-homology sphere and $\mathrm{Pic}\, L \leqslant \mathrm{Pic}\, K \leqslant 7$, so it is a PL sphere by Theorem 4.3.5. By induction, the link of each face is a PL sphere. $\square$

If we proceed our enumeration inductively, we can use the following method for verifying the PL sphereness of a given weak pseudo-manifold. Let $\mathcal{S}^\circ(n,p)$ denote the set of $\mathbb{Z}_2^n$-colorable seeds of Picard number p and dimension $n-1$, up to isomorphism. Let us suppose that we have obtained all $\mathcal{S}^\circ(k,p)$ for $k<n$ and $p \leqslant 4$. Given any $\mathbb{Z}_2^n$-colorable weak pseudo-manifold K, we apply the following procedure to check if it is a PL sphere:

1. Check if the $\mathbb{Z}_2$-Betti numbers of K are the ones of a sphere, namely $(1,0,\dots,0,1)$.
2. For every vertex v of K, let $K_v = \mathrm{lk}_K(v)$, and let L_v be the seed that K_v is obtained from. Since the PL sphereness property is invariant under the wedge operation, we need to check for every v that L_v is isomorphic to a representative in $\mathcal{S}^\circ(k,p)$, for some $p \leqslant 4$ and $k<n$. For this purpose, we use the isomorphism-checking method we provided in Section 4.3.2.

We now have the tools for checking:

— the seedness condition on a simplicial complex with Proposition 4.3.1,

— the existence of an isomorphism between two simplicial complexes in Section 4.3.2, and

— the PL sphereness of a weak pseudo-manifold of Picard number 4 with Lemma 4.3.6.

4.4 Toric colorable PL spheres of Picard number four

In this section, we focus on enumerating all $(n-1)$-dimensional toric colorable seeds of Picard number 4.

4.4.1 A first intuitive procedure

One could intuitively try to find all PL spheres, and compute their (real) Buchstaber numbers. However, it is hopeless when we consider high dimensions. We could obtain results up to $n=6$ by applying either Algorithm 4.2.2 or other known methods such as lexicographic enumeration [130], but it seems to take too long to finish for bigger n.

Remark 4.4.1. Up to isomorphism, one can compute the numbers of PL spheres of dimension $n-1$ and seeds of Picard number 4 up to $n=6$, and their real Buchstaber numbers $s_\mathbb{R}$ as follows.

n	2	3	4	5	6
PL spheres	1	5	39	337	6257
$s_\mathbb{R}=4$	1	5	37	281	2353
$s_\mathbb{R}=3$	0	0	2	56	3904
seeds	1	4	23	194	4237
$s_\mathbb{R}=4$	1	4	21	142	733
$s_\mathbb{R}=3$	0	0	2	52	3504

Since we focus on $\mathbb{Z}_2^n$-colorable seeds, we use a different approach in all that follows.

4.4.2 Enumeration for $n \leqslant 10$

In this subsection, we enumerate all $(n-1)$-dimensional $\mathbb{Z}_2^n$-colorable seeds on $[m]$ of Picard number 4 for $n \leqslant 10$.

Suppose that a $\mathbb{Z}_2^n$-colorable seed supports an IDCM. We first investigate the combinatorial structure of the IDCM itself.

A *matroid* M is a simplicial complex with the *augmentation property*; for any $\tau, \sigma \in M$ with $|\tau| < |\sigma|$, there exists $x \in \sigma \setminus \tau$ such that $\tau \cup \{x\} \in M$. The *dual matroid* $\overline{M}$ of M is defined on the same vertex set as M, and its facets are the complements of each facets of M, which are called the *cofacets* of M. For a full row-rank $n \times m$ matrix $\lambda^{\mathbb{R}}$ over $\mathbb{Z}_2$, the simplicial complex $M_{\lambda^{\mathbb{R}}}$, whose facets are the sets of column indexes of n linearly independent columns of $\lambda^{\mathbb{R}}$, forms a matroid. This matroid is called the *binary matroid* associated with $\lambda^{\mathbb{R}}$. Therefore, K supports a mod 2 characteristic map $\lambda^{\mathbb{R}}$ if and only if K is a subcomplex of $M_{\lambda^{\mathbb{R}}}$. By linear Gale duality [69], the dual matroid $\overline{M_{\lambda^{\mathbb{R}}}}$ is equal to $M_{\overline{\lambda^{\mathbb{R}}}^t}$. We can easily verify the following proposition using the definitions of $M_{\lambda^{\mathbb{R}}}$ and $M_{\overline{\lambda^{\mathbb{R}}}^t}$.

Proposition 4.4.2. *Let K be an $(n-1)$-dimensional simplicial complex on $[m]$ and $\overline{\lambda^{\mathbb{R}}}$ an $m \times (m-n)$ matrix over $\mathbb{Z}_2$ of rank $m-n$. Then K supports $\overline{\lambda^{\mathbb{R}}}$ as a DCM if and only if it is a subcomplex of $\overline{M_{\overline{\lambda^{\mathbb{R}}}^t}} = M_{\lambda^{\mathbb{R}}}$.*

As we recall, reducing the number of facets in the input of Algorithm 4.2.2 leads to a smaller dimension of the mod 2 kernel of the ridge-facet incidence matrix, resulting in faster execution of Algorithm 4.2.2. Proposition 4.4.2 provides us with a smaller set of facets, as desired.

Furthermore, we leverage the upper bound theorem for facets of PL spheres, see [128]. According to this theorem, the number of facets of an $(n-1)$-dimensional simplicial sphere of Picard number 4 is less than or equal to the number of facets of the cyclic n-polytope $C^n(n+4)$ with $n+4$ vertices. It is known that the number of facets of $C^n(n+4)$ is

$$f_{n-1}(C^n(n+4)) = \binom{n+4-\lceil \frac{n}{2} \rceil}{4} + \binom{n+3-\lfloor \frac{n}{2} \rfloor}{4},$$

see [26] for example.

This condition is represented by the affine function $g(K) = f_{n-1}(C^n(n+4)) - \|K\|_1 + 1$, where $\|K\|_1$ is the 1-norm of the vector K, which corresponds to the number of facets of K. Let $\overline{\lambda^{\mathbb{R}}}\colon [m] \longrightarrow \mathbb{Z}_2^4$ be an injective map, and denote by $\mathcal{F}(\lambda^{\mathbb{R}}) = \mathcal{F}(M_{\lambda^{\mathbb{R}}})$ the set of facets of the associated binary matroid. Algorithm 4.2.2, with inputs $\mathcal{F}(\lambda^{\mathbb{R}})$ and the affine function g, outputs the set of all weak pseudo-manifolds that support $\overline{\lambda^{\mathbb{R}}}$ and satisfy the upper bound theorem.

At first glance, it might seem necessary to run the algorithm on each of the $\binom{11}{n} \times n!$ injective maps $\overline{\lambda^{\mathbb{R}}}$, even if we fix $\begin{bmatrix} \overline{\lambda^{\mathbb{R}}}(n+1) & \overline{\lambda^{\mathbb{R}}}(n+2) & \overline{\lambda^{\mathbb{R}}}(n+3) & \overline{\lambda^{\mathbb{R}}}(n+4) \end{bmatrix} = I_4$. However, we can significantly reduce this large number of cases by observing that many injective maps yield the same outputs up to isomorphism.

Let $\Lambda(n,p)$ be the set of all $(n+p) \times p$ matrices over $\mathbb{Z}_2$ of the form $\begin{bmatrix} M \\ I_p \end{bmatrix}$, with no repeated rows. We consider the product of two symmetric groups $\mathfrak{S}_n \times \mathfrak{S}_p$ which acts on $\Lambda(n,p)$ as follows: $\left(\begin{bmatrix} M \\ I_p \end{bmatrix}, (s,t) \right) \mapsto \begin{bmatrix} P_s^t M P_t \\ I_p \end{bmatrix}$, where P_s and P_t are column permutation matrices corresponding to permutations s and t. Let us call each element of $\Lambda(n,p)/\mathfrak{S}_n \times \mathfrak{S}_p$ an *IDCM orbit*.

Proposition 4.4.3. *For $(s,t) \in \mathfrak{S}_n \times \mathfrak{S}_p$, there is an isomorphism between the binary matroids associated to $\overline{\lambda^{\mathbb{R}}} \in \Lambda(n,p)$ and $\overline{\lambda^{\mathbb{R}}} \circ (s,t) \in \Lambda(n,p)$.*

Proof. We partition the vertex set $[m]$ into $V_1 = [n]$ and $V_2 = \{n+1, n+2, \dots, n+p\}$. It is evident that the matrix $\begin{bmatrix} P_s^t M \\ I_p \end{bmatrix}$ preserves the same non-singularity information on cofacets, with the vertices in V_1 relabeled according to permutation s. Additionally, since P_t is invertible, we have $M_{\overline{(\lambda^{\mathbb{R}}P_t)^t}} = M_{\overline{\lambda^{\mathbb{R}}}^t}$. Then, applying t to V_2 in the equation

$$\begin{bmatrix} P_s^t M P_t \\ I_p \end{bmatrix} = \begin{bmatrix} P_s^t M \\ P_t^t I_p \end{bmatrix} P_t,$$

yields the same non-singularity conditions for $\begin{bmatrix} P_s^t M \\ I_p \end{bmatrix}$ and $\begin{bmatrix} P_s^t M P_t \\ I_p \end{bmatrix}$. □

Let $\Lambda^\circ(n,4) \subseteq \Lambda(n,4)$ be a set containing one representative per IDCM orbit. By Proposition 4.4.3, it is enough to input Algorithm 4.2.2 with $\mathcal{F}(\lambda^{\mathbb{R}})$ for all $\lambda^{\mathbb{R}} \in \Lambda^\circ(n,4)$. Table 4.1 displays how efficient our method is. It provides the dimension of the kernel of the ridge-facet incidence matrix $A\left(\binom{[m]}{n}\right)$ and the size of the set $\mathcal{X}\left(\binom{[m]}{n}\right)$ which represents the number of element in its kernel for which we should verify whether they are weak pseudo-manifolds, together with $\max_{\overline{\lambda^{\mathbb{R}}}}(\dim\ker A(\mathcal{F}(\lambda^{\mathbb{R}})))$ and $\max_{\overline{\lambda^{\mathbb{R}}}} |\mathcal{X}(\mathcal{F}(\lambda^{\mathbb{R}}))|$. The number of IDCM orbits of $\Lambda(n,4)$ and the computation time of the call of Algorithm 4.2.2 at line 5 of Algorithm 4.4.4 are also provided. This demonstrates that our reductions enable computability of the problem, for example with $n = 10$:

— the choice of the convenient basis made in Section 4.2.1 reduces the number of cases from $2^{286} \simeq 1e86$ to $5e74$ for the set of facets $\binom{[m]}{n}$, and from $2^{56} \simeq 7e16$ to $4e14$ for $\mathcal{F}(\lambda^{\mathbb{R}})$, and

— taking into account that $\mathbb{Z}_2^n$-colorable seeds are subcomplexes of the binary matroid associated to an IDCM divides the number of cases to compute by a factor of 10^{60}.

n	2	3	4	5	6	7	8	9	10	11
$\dim\ker A\left(\binom{[m]}{n}\right)$	10	20	35	56	84	120	165	220	286	364
$\lvert\mathcal{X}\left(\binom{[m]}{n}\right)\rvert$	352	2e5	1e9	3e14	7e18	8e21	3e31	4e57	5e74	2e93
Number of IDCM orbits	7	16	28	35	35	28	16	7	3	1
$\max_{\overline{\lambda^{\mathbb{R}}}}(\dim\ker A(\mathcal{F}(\lambda^{\mathbb{R}})))$	7	13	21	24	28	34	42	48	56	64
$\max_{\overline{\lambda^{\mathbb{R}}}} \lvert\mathcal{X}(\mathcal{F}(\lambda^{\mathbb{R}}))\rvert$	56	3e3	5e5	1e6	2e7	9e8	1e11	3e12	4e14	4e16
Time spent for one orbit	1ms	10ms	0.1s	0.6s	1.3s	3m	15m	2h	12d	**3y**

Table 4.1 – Data table for Picard number 4 and $n = 2, \dots, 11$. The time spent refers to Algorithm 4.2.2 running on an Nvidia Quadro A5000. The time written in bold in the case $n = 11$ is an estimation.

Algorithm 4.4.4. The full procedure for obtaining every $(n-1)$-dimensional seed of Picard number 4 supporting an IDCM is as follows.

Input: Integer $n \geqslant 2$.

Output: The set $\mathcal{K}^\circ(n,4)$ of $(n-1)$-dimensional seeds of Picard number 4 supporting an IDCM up to isomorphism.

1: **Procedure** GetIDCM-ColorableSeedsPic4(n)
2: Compute $\Lambda^\circ(n,4) \subseteq \Lambda(n,4)$ a set containing one representative for each IDCM orbit.
3: $\mathcal{K}(n,4) \leftarrow \varnothing$
4: **for all** $\lambda^{\mathbb{R}} \in \Lambda^\circ(n,4)$ **do**
5: $\mathcal{K}(\lambda^{\mathbb{R}}) \leftarrow$ output of Algorithm 4.2.2 with inputs $\mathcal{F}(\lambda^{\mathbb{R}})$ and $\mathcal{G} = \{g\}$, with $g(K) = f_{n-1}(C(n,n+4)) - \|K\|_1 + 1$

6: $\mathcal{K}(n,4) \leftarrow \mathcal{K}(n,4) \cup \mathcal{K}(\lambda^{\mathbb{R}})$
7: **end for**
8: **for all** $K \in \mathcal{K}(n,4)$ **do**
9: **if** $\mathrm{Pic}(K) < 4$ **then** discard K
10: **end if**
11: **if** K does not satisfy the seedness condition **then** discard K
12: **end if**
13: **end for**
14: Select $\mathcal{K}^\circ(n,4) \subseteq \mathcal{K}(n,4)$ with one representative K up to isomorphism
15: **for all** $K \in \mathcal{K}^\circ(n,4)$ **do**
16: **if** K is not a PL sphere **then** discard K
17: **end if**
18: **end for**
19: **end Procedure**

Running Algorithm 4.4.4 for $n \leqslant 10$ gives Table 4.2.

n	2	3	4	5	6	7	8	9	10
$\lvert\mathcal{K}(n,4)\rvert$ at line 7	90	1119	20383	79877	322837	503624	469445	224854	99374
$\lvert\mathcal{K}(n,4)\rvert$ at line 13	22	578	13679	47012	204714	310217	305280	140933	57956
$\lvert\mathcal{K}^\circ(n,4)\rvert$ at line 15	2	5	49	256	1791	2194	1401	381	56
$\lvert\mathcal{K}^\circ(n,4)\rvert$ at line 19	1	4	20	142	733	1190	776	243	39

Table 4.2 – The output of Algorithm 4.4.4 for $n \leqslant 10$.

Now, all that remains are the seeds that do not admit any IDCM. We show there actually remains a single one not outputted by Algorithm 4.4.4.

From Theorem 4.3.4, they must be suspensions. Let $L = \partial[v,w] * K$ be the suspension of an $(n-2)$-dimensional simplicial complex K, and suppose that L is $\mathbb{Z}_2^n$-colorable. We may assume that a characteristic map $\lambda^{\mathbb{R}}$ over L satisfies $\lambda^{\mathbb{R}}(v) = \begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}^t$. Then for any facet $\{v\} \cup \{v_1, \ldots, v_{n-1}\}$ of L, the $(1,v)$ minor of the matrix $\begin{bmatrix}\lambda^{\mathbb{R}}(v) & \lambda^{\mathbb{R}}(v_1) & \cdots & \lambda^{\mathbb{R}}(v_{n-1})\end{bmatrix}$ is equal to 1. This implies that $\mathrm{lk}_L(1) = K$ is $\mathbb{Z}_2^{n-1}$-colorable. Hence, the suspension operation preserves $\mathbb{Z}_2^n$-colorability while also preserving seedness, and increases the Picard number by one. Therefore, it is sufficient to consider the suspensions of the seeds of Picard number 3.

The three $\mathbb{Z}_2^n$-colorable seeds of Picard number 3 are the boundaries of a pentagon, a 3 dimensional cross polytope, and a cyclic polytope $C^4(7)$ [67].

The suspension of a pentagon and a cyclic polytope both support an IDCM and have been obtained in Table 4.2 for $n = 3$ and 5. Finally, the boundary of a cross polytope does not support any IDCM, but does support a DCM, so we add it to the result $\mathcal{K}^\circ(4,4)$ in Table 4.2.

Theorem 4.4.5. *Up to isomorphism, the number of $\mathbb{Z}_2^n$-colorable seeds of dimension $n-1$ and Picard number 4 for $n \leqslant 10$ is as follows.*

n	2	3	4	5	6	7	8	9	10
$\mathbb{Z}_2^n$-colorable seeds	1	4	$20+1$	142	733	1190	776	243	39

4.4.3 Enumeration for $n = 11$

As shown in Table 4.1, the time complexity of the extreme case $n = 11$ remains too long. To address this, we leverage the results obtained from the dimension just below to construct the seeds for this extreme case.

Let K be a $\mathbb{Z}_2^{11}$-colorable seed on $\{1, 2, \dots, 15\}$ of dimension 10 ($n = 11$). We know that the link of the vertex 15 has Picard number $\leqslant 4$ and is a $\mathbb{Z}_2^{10}$-colorable seed, which we have already enumerated. We construct all $\mathbb{Z}_2^{11}$-colorable seeds of dimension 10 from the $\mathbb{Z}_2^{10}$-colorable ones of dimension 9. Firstly, if K has only vertices whose links have Picard numbers at most 2, then K is the boundary of a product of simplices [82], and therefore not a seed. Suppose that the link of 15 has Picard number 3. Since there is no 9-dimensional seed of Picard number 3, it implies that the link of vertex 15 is not a seed. By the following lemma, we can identify another vertex of K whose link has Picard number 4.

Lemma 4.4.6. *Let K be a seed of Picard number 4. Assume that K has a vertex v such that $\mathrm{Pic}(\mathrm{lk}_K(v)) = 3$ and there exist two vertices v_1 and v_2 of $\mathrm{lk}_K(v)$ such that every facets of $\mathrm{lk}_K(v)$ contains either v_1 or v_2. Then, there is a vertex of K whose link has Picard number 4.*

Proof. Let $\{v_1\} \cup \sigma$ be a facet without v_2. There is one more facet containing σ since it is a ridge of the PL sphere $\mathrm{lk}_K(v)$. By assumption, it must be $\{v_2\} \cup \sigma$. That shows every vertex in $\mathrm{lk}_K(v)$ forms an edge with both v_1 and v_2.

Let w be the vertex not in $\mathrm{lk}_K(v)$. If $v_1 \in \mathrm{lk}_K(w)$, then $\mathrm{lk}_K(v_1)$ has Picard number 4. If both $v_1, v_2 \notin \mathrm{lk}_K(w)$, then $\mathrm{lk}_K(w)$ is an $(n-2)$-dimensional PL sphere with n vertices. This means that $\mathrm{lk}_K(w) = \partial\Delta_{n-1}$. Then if w' is a vertex of $\mathrm{lk}_K(w)$ other than w, then $\mathrm{Pic}(\mathrm{lk}_K(w')) = 4$. □

That implies any $\mathbb{Z}_2^{11}$-colorable seed of Picard number 4 has a vertex whose link also has Picard number 4, which we relabel as vertex 15.

Before we apply Algorithm 4.4.4 for this case, we need some preparation as follows. We firstly select an injective map $\bar{\mu}\colon \{1, \dots, 14\} \to \mathbb{Z}_2^4$ and choose a 9-dimensional PL sphere L that supports $\bar{\mu}$. We see L as the link of the vertex 15 in some $\mathbb{Z}_2^{11}$-colorable seed K supporting some IDCM $\overline{\lambda^{\mathbb{R}}}$ with the restriction $\overline{\lambda^{\mathbb{R}}}|_{\{1,\dots,14\}} = \bar{\mu}$. Since $\left|\mathbb{Z}_2^4 \setminus \{0\}\right| = 15$, once $\bar{\mu}$ is chosen, $\overline{\lambda^{\mathbb{R}}}$ is uniquely determined. There are 114 9-dimensional $\mathbb{Z}_2^{10}$-colorable PL spheres that support an IDCM, among which 39 are seeds and 75 are non-seeds. They can be obtained from Algorithm 4.4.4 by skipping the step that discards the non-seeds, for instance.

Let $\hat{L}$ be the simplicial complex $\{\sigma \cup \{15\} \mid \sigma \in L\}$. All PL spheres K having its vertex 15 whose link is L contains $\hat{L}$. That provides the following conditions on the components of $K \in \mathbb{Z}^M$:

1. for all $\hat{L}_j = 1$, $K_j = 1$, and
2. for all $\hat{L}_j = 0$ with $\hat{L}_j \ni \{15\}$, $K_j = 0$.

We will denote by I and J the set of indexes of the facets satisfying Condition (1), respectively Condition (2). After reordering the rows of B, the two conditions appear as follows

$$BX = \begin{bmatrix} B_I \\ B_J \\ B_{[M]\setminus(I\cup J)} \end{bmatrix} X = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \star \end{bmatrix}. \tag{4.4.1}$$

A mod 2 Gaussian elimination process on the columns of B gives a column-reduced echelon form $\widetilde{B}$ which yields another set of generators for the mod 2 kernel of A. Denote by s_I and s_J the maximal index of non-zero column of $\widetilde{B}_I$ and of $\widetilde{B}_J$, respectively. To respect conditions (1) and

(2), we need

$$x_t = \begin{cases} 1 & t = 1, \dots, s_I \\ 0 & t = s_I + 1, \dots, s_J \\ \star & \text{otherwise} \end{cases},$$

for $X = (x_1, \dots, x_M)^t$. If no such X satisfy this conditions, then there is no $\mathbb{Z}_2^{11}$-colorable seed K that supports $\lambda^{\mathbb{R}}$ and whose link of the vertex 15 is L.

Applying Algorithm 4.4.4 with a revised version of the initialization of Algorithm 4.2.2 that takes into account Condition (4.4.1) on the entries of X yields the following.

Theorem 4.4.7. *There are exactly* 4 $\mathbb{Z}_2^{11}$*-colorable* 10*-dimensional seeds of Picard number* 4*.*

4.4.4 Toric colorability

Remember that it is enough to check whether each seed among the $\mathbb{Z}_2^n$-colorable ones supports a characteristic map for obtaining all the toric colorable seeds of Picard number 4.

Let K be a $\mathbb{Z}_2^n$-colorable seed of Picard number 4, and $\lambda^{\mathbb{R}}$ a mod 2 characteristic map over K. If we regard each image vector of $\lambda^{\mathbb{R}}$ as a $\{0,1\}$-vector of $\mathbb{Z}^n$, and denote by λ the obtained map, then λ is not necessarily a characteristic map over K. Therefore, we change some 1's in the image vectors of λ to -1's until it becomes a characteristic map over K. Brute-forcing this method provides at least one characteristic map supported by every $\mathbb{Z}_2^n$-colorable seed we enumerated. The toric colorability is thus equivalent to the $\mathbb{Z}_2^n$-colorability for PL spheres of Picard number 4. This yields the full theorem.

4.5 Application to the normalized space of rational curves on toric manifolds of Picard number four

This section is devoted to answering a question of Chen, Fu, and Hwang in [32] and assume the reader is familiar with it. We also refer to [100, 94] for more details about rational curves on algebraic varieties. Let X be a toric manifold. For an irreducible component $\mathcal{K}$ of the normalized space of rational curves on X, denote by $\rho\colon \mathcal{U} \longrightarrow \mathcal{K}$ and $\mu\colon \mathcal{U} \longrightarrow X$ the associated universal family morphisms. The irreducible component $\mathcal{K}$ is called a *minimal component* if μ is dominant and for a general point $x \in X$, the variety $\mu^{-1}(x)$ is complete. Members of such $\mathcal{K}$ are called *minimal rational curves* and the *degree* of $\mathcal{K}$ is defined by the degree of the intersection of the anti-canonical divisor of X with any member in $\mathcal{K}$.

Recall that X is characterized by a complete non-singular fan with its underlying simplicial complex K on $[m]$ and its primitive ray vectors $\lambda(i)$, for $i \in [m]$. In particular, λ is a characteristic map over K. Conversely, if a characteristic map λ over K gives a fan with underlying simplicial complex K, then λ is called *fan-giving*.

For each minimal non-face $\{v_1, v_2, \dots, v_k\}$ of K, the set $\{\lambda(v_1), \lambda(v_2), \dots, \lambda(v_k)\}$ is called a *primitive collection* of (K, λ).

Theorem 4.5.1. *[32, Proposition 3.2] Let X be a toric manifold of complex dimension n, and let (K, λ) represent its associated fan. The minimal components of degree k on X bijectively correspond to primitive collections $\{\lambda(v_1), \lambda(v_2), \dots, \lambda(v_k)\}$ of (K, λ) such that $\lambda(v_1) + \cdots + \lambda(v_k) = 0$.*

Let $V = \{\lambda(v_1), \lambda(v_2), \dots, \lambda(v_k)\}$ and $W = \{\lambda(w_1), \dots, \lambda(w_l)\}$ be two primitive collections that correspond to two minimal components. Assume that they intersect, so without loss of generality, $\lambda(v_k) = \lambda(w_l)$. Then

$$\lambda(v_1) + \cdots + \lambda(v_{k-1}) = -\lambda(v_k) = -\lambda(w_l) = \lambda(w_1) + \cdots + \lambda(w_{l-2}).$$

This means that the two cones generated by $V \setminus \{v_k\}$ and $W \setminus \{w_l\}$ are the same. Hence for two primitive collections V and W corresponding to minimal components, there are only two possibilities: either $V = W$, or $V \cap W = \varnothing$. Using this property and the previous theorem one obtains the following inequality.

Proposition 4.5.2. *[32, Proposition 3.5] Let X be a toric manifold of complex dimension n and Picard number p. Then*

$$\sum_{k=0}^{n-1} n_k(k+2) \leqslant n + p, \tag{4.5.1}$$

where n_k is the number of minimal components in the normalized space of rational curves on X of degree $k+2$.

We consider the fan associated to X, represented by a pair (K, λ). Through a direct interpretation of inequality (4.5.1), equality holds if and only if there is a partition P of the vertex set of K such that for each $\sigma = \{v_1, \ldots, v_k\} \in P$, σ is a minimal non-face of K, and $\lambda(v_1) + \cdots + \lambda(v_k) = 0$. Let us call such partition *optimal.* The rest of this section is devoted to finding in which cases there exists such an optimal partition for K of Picard number $\leqslant 4$ that supports a fan-giving characteristic map.

Firstly, observe that the left multiplication of a characteristic matrix by an invertible matrix does not affect whether it has an optimal partition. Such two matrices are called Davis-Januszkiewicz equivalent (or simply *D-J equivalent*). Hence, we suppose that the first n columns of an $n \times m$ characteristic matrix over K form the $n \times n$ identity matrix with the assumption that $\{1, 2, \ldots, n\}$ is a facet of K.

For a simplicial complex K, assume that the following two matrices are (mod 2) characteristic maps over K;

$$\begin{aligned} \lambda &= \begin{bmatrix} 1 & \mathbf{0} & \mathbf{a} \\ \mathbf{0} & I_{n-1} & A \end{bmatrix}, \text{ and} \\ \mu &= \begin{bmatrix} 1 & \mathbf{0} & \mathbf{b} \\ \mathbf{0} & I_{n-1} & A \end{bmatrix}. \end{aligned} \tag{4.5.2}$$

Then, using the notations introduced in [49], the matrix

$$\lambda \wedge_1 \mu = \begin{bmatrix} 1 & 0 & \mathbf{0} & \mathbf{a} \\ 0 & 1 & \mathbf{0} & \mathbf{b} \\ \mathbf{0} & \mathbf{0} & I_{n-1} & A \end{bmatrix} \tag{4.5.3}$$

is a (mod 2) characteristic map over $\mathrm{wed}_1(K)$ if it satisfies the (mod 2) non-singularity condition. For any other vertex v, one can construct a (mod 2) characteristic map over $\mathrm{wed}_v(K)$ from two (mod 2) characteristic maps over K similarly.

Theorem 4.5.3. *[44] For a vertex v of a simplicial complex K, every (mod 2) characteristic map over $\mathrm{wed}_v(K)$ is of the form $\lambda \wedge_v \mu$ for two (mod 2) characteristic maps λ and μ over K up to D-J equivalence. Moreover, $\lambda \wedge_v \mu$ is fan-giving if and only if both λ and μ are fan-giving.*

Remark 4.5.4. Let v be a vertex of K. Notice that by taking any two (mod 2) characteristic maps λ and μ over K we cannot always construct the matrix "$\lambda \wedge_v \mu$". However, for a single (mod 2) characteristic map over K, the matrix $\lambda \wedge_v \lambda$ can always be constructed. Such (mod 2) characteristic map over $\mathrm{wed}_v(K)$ is called the *canonical extension* of λ at v. For more details on the compatibility of the operation $\wedge_v$ in the mod 2 case, see [49], or Chapter 5.

Lemma 4.5.5. *For a simplicial complex K and a vertex v of K, let $\Lambda = \lambda \wedge_v \mu$ be a fan-giving characteristic map over* $\mathrm{wed}_v(K)$*, where λ and μ are characteristic maps over K. Then* $(\mathrm{wed}_v(K), \Lambda)$ *has an optimal partition if and only if both* (K, λ) *and* (K, μ) *have optimal partitions.*

Proof. Without loss of generality, it is enough to consider the case $v = 1$. Then we can additionally assume that characteristic maps are of the forms (4.5.2) and (4.5.3). Then for a subset $\{v_1, \dots, v_k\}$ of the vertex set of K, $\Lambda(v_1) + \cdots + \Lambda(v_k) = 0$ if and only if the sum is zero component-wise. Hence $\Lambda(v_1) + \cdots + \Lambda(v_k) = 0$ if and only if $\lambda(v_1) + \cdots + \lambda(v_k) = 0$ and $\mu(v_1) + \cdots + \mu(v_k) = 0$. □

By Theorem 4.5.3 and Lemma 4.5.5, it is enough to investigate only seeds of Picard number p to determine which fan of Picard number p has an optimal partition. Then, by Theorem 4.1.2, we obtain the following corollary.

Corollary 4.5.6. *A PL sphere K of Picard number $p \leqslant 4$ supports a fan-giving characteristic map λ that has an optimal partition if and only if K is achieved by a sequence of wedge operations from the boundary of*

— *a* 1*-simplex if $p = 1$,*

— *a square if $p = 2$,*

— *a* 3*-dimensional cross polytope if $p = 3$, and*

— *either a hexagon or a* 4*-dimensional cross polytope if $p = 4$.*

Moreover, each of the listed seeds supports a unique fan-giving characteristic map with an optimal partition, and thus λ is obtained by sequential canonical extensions.

Before proving Corollary 4.5.6, we need some preliminary results about the join of two simplicial complexes K and L on $\{1, 2, \dots, m_1\}$ and $\{m_1+1, m_1+2, \dots, m_2\}$, respectively. Any minimal non-face of $K * L$ is a minimal non-face of either K or L. In addition, a (mod 2) characteristic map λ over $K * L$ can be represented as

$$\lambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix},$$

where λ_{11} is a characteristic map over K, and λ_{22} is a characteristic map over L. Furthermore, by D-J equivalence, if $\dim(K) = n_1 - 1$, then we can assume that the first n_1 columns of λ_{11} form an identity matrix, and the first n_1 columns of λ_{21} are zeros. A similar argument holds for λ_{12} and λ_{22}.

Proof of Corollary 4.5.6. We first consider the boundary of any cross polytope. Recall that the suspension of K is the join of a 0-sphere S^0 and K. Then any characteristic map λ over $S^0 * S^0$ is of the form

$$\begin{bmatrix} 1 & \pm 1 & 0 & b \\ 0 & a & 1 & \pm 1 \end{bmatrix}$$

up to D-J equivalence if the vertex set of each S^0's are $\{1, 2\}$ and $\{3, 4\}$. Hence $(S^0 * S^0, \lambda)$ has an optimal partition if and only if

$$\lambda = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

up to D-J equivalence. Since $S^0 * \cdots * S^0$ has minimal non-faces $\{1,2\},\{3,4\},\ldots$, a similar argument holds, so $(S^0 * \cdots * S^0, \lambda)$ has an optimal partition if and only if λ is a block diagonal matrix whose block diagonal elements are all $\begin{bmatrix} 1 & -1 \end{bmatrix}$ up to D-J equivalence. Note that any toric colorable seed of Picard number 1 is S^0, and any of Picard number 2 is $S^0 * S^0$.

For Picard number 3, there are two toric colorable seeds that are not the boundary of a 3-dimensional cross polytope:

1. the boundary of a pentagon.
2. the boundary of a 4-dimensional cyclic polytope $C^4(7)$ with 7 vertices.

For (1), one can check that there is no partition of the vertex set consisting of minimal non-faces. For (2), there is no fan-giving characteristic map from [44].

Finally, for Picard number 4, we use the list of toric colorable seeds obtained in Theorem 4.1.2. Similarly to the classification of toric colorable PL spheres, we approach it from the mod 2 characteristic map perspective. Suppose that P is an optimal partition for (K,λ). Then for any $\{v_1,\ldots,v_k\} \in P$, the sum $\lambda(v_1)+\cdots+\lambda(v_k)$ is also zero in mod 2. For a mod 2 characteristic map $\lambda^{\mathbb{R}}$ over K, we call a partition P of $[m]$ *weakly optimal* if P satisfies the condition of optimal partition with a mod 2 characteristic map $\lambda^{\mathbb{R}}$ instead of a characteristic map. Since K has finitely many mod 2 characteristic maps, we can investigate all possibilities. For an $(n-1)$-dimensional regular seed, there is no partition consisting of minimal non-faces if n is 10 or 11. Moreover, the boundary of the hexagon is the only Picard number 4 toric colorable regular seed which has a weakly optimal partition. More precisely, consider the boundary K of the hexagon whose facets are $\{1,2\},\{1,6\},\{2,3\},\{3,4\},\{4,5\},\{5,6\}$. Then there are four partitions, as follows: $P_1 = \{\{1,3\},\{2,5\},\{4,6\}\}$, $P_2 = \{\{1,4\},\{2,6\},\{3,5\}\}$, $P_3 = \{\{1,5\},\{2,4\},\{3,6\}\}$, and $P_4 = \{\{1,4\},\{2,5\},\{3,6\}\}$. Suppose that P_1 is an optimal partition of (K,λ). We can choose a D-J class with $\lambda(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\lambda(2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then by optimality of P_1, $\lambda(3) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $\lambda(5) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. By fan-givingness of λ, $\lambda(4) = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$. Again, by optimality of P_1, $\lambda(6) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, but λ does not yield a fan. Hence there is no λ such that P_1 is an optimal partition for (K,λ). The partitions P_2 and P_3 are the same as P_1 since they are obtained after rotating the labels of the hexagon. For P_4, by using optimality and fan-givingness, one can similarly show that it is an optimal partition of (K,λ) if and only if $\lambda = \begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 & -1 \end{bmatrix}$.

There are in addition three suspended seeds of Picard number 4 other than the boundary of a 4-dimensional cross polytope:

1. the suspension of the boundary of a pentagon, and
2. the suspension of the boundary of $C^4(7)$.

For these cases, by construction of the suspension, there is no optimal partition from similar reasons as for the Picard number 3 case. $\square$

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5 The mod 2 puzzle algorithm

"Everywhere, in whatever realm of life, whether among its callous, coarsely impoverished and messily moldering lower ranks, or among its monotonously gelid and tediously tidy upper strata, everywhere, if but once, a person will encounter a phenomenon on his journey that is unlike anything he has chanced to see heretofore and that, at least once will awake in him a feeling unlike any he is fated to feel for the rest of his life."

Nikolai Gogol — Dead Souls

This chapter is based on a joint work with Suyoung Choi published in the Pacific Journal of Mathematics [49]. Within this chapter, we focus on mod 2 characteristic maps and will write them as λ instead of $\lambda^{\mathbb{R}}$, and they will be referred to as "characteristic maps" (we omit the "mod 2"). We describe an efficient algorithm for constructing every mod 2 characteristic maps over a simplicial complex. This algorithm is based on an idea from Choi and Park, using the simplicial wedge operation, which they called the "puzzle method" [46].

5.1 Introduction

Let K be a pure simplicial complex of dimension $n-1$ on the vertex set V, where n is a positive integer. A map $\lambda\colon V \to \mathbb{Z}_2^n$ is called a (mod 2) *characteristic map* over K if it satisfies the following *non-singularity condition*: if $\{i_1, \ldots, i_n\} \in K$, then $\{\lambda(i_1), \ldots, \lambda(i_n)\}$ is linearly independent. Two characteristic maps over K are said to be *DJ-equivalent* if one can be obtained from the other by changing basis of $\mathbb{Z}_2^n$. Denote by $\mathrm{CM}(K)$ the DJ-equivalence classes of characteristic maps over K. It is known that $\mathrm{CM}(K)$ provides the classification of some important classes of manifold. For instance, if K is polytopal, $\mathrm{CM}(K)$ is the classification of small covers introduced in [60] whose orbit space is combinatorially isomorphic to K.

One important question in toric topology is "how to find $\mathrm{CM}(K)$ for a given K?". There have been many attempts to list up all mod 2 characteristic maps over some specific (polytopal) PL spheres such as a dodecahedron [75], prisms [29], cubes [36], and polytopes with a few vertices [67], [44].

For this purpose, one remarkable algorithm was introduced by Garrison and Scott [75]. This algorithm produces $\mathrm{CM}(K)$ for any pure simplicial complex K. However, since it is a branch-and-bound algorithm, its complexity becomes relatively bad when increasing either the dimension $n-1$ or the number of vertices m, and, hence, it can be used for neither high dimensional cases nor for equipping many vertices.

Another remarkable algorithm was provided after by Choi and Park ([44], [46]). In order to present their method, we need to introduce some notations. The *Picard number* $\mathrm{Pic}(K)$ of K is defined as $\mathrm{Pic}(K) = m - n$, where $m = |V|$. For $v \in V$, the *wedge* of K at v is the simplicial complex on $V \cup \{v_1, v_2\} \setminus \{v\}$ defined by

$$\mathrm{wed}_v(K) := (I * \mathrm{lk}_K(v)) \cup (\partial I * K \setminus \{v\}), \tag{5.1.1}$$

where I is the 1-simplex with vertices $\{v_1, v_2\}$, $K \setminus F := \{\sigma \in K \mid F \not\subset \sigma\}$, for a face $F \in K$, and $\mathrm{lk}_K(v)$ is the link of K at the vertex v. Set $V = [m] := \{1, \ldots, m\}$. More generally, for an m-tuple $J = (j_1, \ldots, j_m)$ of positive integers, denote by $K(J)$ the *wedged simplicial complex* obtained from K after performing $j_v - 1$ wedges on each vertex $v \in V$. It should be noted that a wedge operation preserves the Picard number. If K is not obtained from a lower dimensional simplicial complex by a wedge operation, then K is called a *seed*. Any simplicial complex K which is not a seed can always be represented as a wedged simplicial complex $L(J)$ with L being a seed.

Choi and Park's method relies on obtaining as a first step $\mathrm{CM}(\mathrm{wed}_v(K))$ from $\mathrm{CM}(K)$. This process can be interpreted as putting stones on a certain board following specific rules, accordingly this method is called *the puzzle method*. A puzzle is said to be *realizable* if all stones of the board respect the rules. For a seed L, we firstly prepare $\mathrm{CM}(L)$. Its low dimension makes the Garrison and Scott algorithm usable. For a wedged simplicial complex $K = L(J)$ and $J = (j_1, \ldots, j_m)$, one can construct $\mathrm{CM}(K)$ from $\mathrm{CM}(L)$ by applying a puzzle method inductively on $j_1 + \cdots + j_m$. In many ways this method is quite powerful since it can handle plenty of interesting cases such as wedges of polygons [47].

Considering an algorithmic viewpoint, the puzzle method would accelerate the process for constructing $\mathrm{CM}(L(J))$ for L a seed and would be faster than computing it directly by the Garrison and Scott algorithm. In addition, it was shown in [46] that, for a fixed Picard number p, the number of seeds L with $\mathrm{Pic}(L) = p$ satisfying $\mathrm{CM}(L) \neq \varnothing$ is finite. More precisely, $n \leqslant 2^p - p$. In other words, when p is small and the dimension is high, then a PL sphere K is either a wedged PL sphere or it does not support any mod 2 characteristic map. Thus, for a small Picard number, an algorithm based on the puzzle method would be a fine upgrade of the Garrison and Scott algorithm. As a by-product, if one can find all seeds of fixed Picard number p that support a mod 2 characteristic map, an algorithm based on the puzzle method enables us to classify some important classes of closed manifolds including small covers and real toric manifolds whose first $\mathbb{Z}_2$-cohomology groups are of rank p.

A straightforward algorithm based on the puzzle method would be to enumerate all possible stone configuration on the board and check if the rules introduced in [46] are respected in each such configuration.

However, the latter rules are somehow complicated and require a lot of preliminary computation. The first rule demands all edges of the board to correspond to a mod 2 characteristic map over $\mathrm{wed}_v(L)$, and this requires to compute $\mathrm{CM}(\mathrm{wed}_v(L))$ for each vertex v of L, i.e. m times. The second one is that every subsquares of the board should correspond to a mod 2 characteristic map over a simplicial complex obtained from L after two consecutive wedge operations, namely $\mathrm{wed}_{v,w}(L)$, and this requires to compute $\mathrm{CM}(\mathrm{wed}_{v,w}(L))$ for each unordered pair of distinct vertices (v, w) of L, i.e. $m(m-1)/2$ times.

Both of these rule computations would be time consuming in any algorithm based on the puzzle method. Moreover, our intuition makes us believe that we could try to fill the board with stones in a constructive way by respecting the rules step by step. Consequently, until now, this method seemed promising but have not been analyzed because of the complexity of these rules.

In this chapter, we zero in on finding a strategy for the puzzle method to be usable as an algorithm. We describe the puzzle method completely in terms of linear algebraic language. It provides explicit formulae for all possible mod 2 characteristic maps over a wedged seed and over a twice wedged seed. These formulae only require the computation of $\mathrm{CM}(L)$ for the seed L and thus make the previously mentioned time consuming computations unnecessary.

Furthermore, we focus on the key concept that a realizable puzzle only requires a few stones positioned on the board to have all of its other stone positions determined [46]. From this we transform the greedy “enumerating all cases” puzzle algorithm onto an elegant constructive and procedural algorithm which uses the edge and square rules previously computed for building a realizable puzzle.

We compute its complexity, and show that it is much faster than a direct use of the Garrison and Scott algorithm on $L(J)$ when we tend to increase the dimension $n-1$.

5.2 Dual characteristic maps

Let K be a PL sphere on $[m] = \{1, \ldots, m\}$ with $\dim(K) = n-1$. A non-singular characteristic map λ over K can be associated to an element of a matrix group

$$\lambda = \left[\begin{array}{c|c|c|c} \lambda(1) & \lambda(2) & \cdots & \lambda(m) \end{array}\right] \in \mathcal{M}(n, m, \mathbb{Z}_2),$$

where $\mathcal{M}(n, m, \mathbb{Z}_2)$ is the set of $n \times m$ $\mathbb{Z}_2$-matrices. The order in which the columns appear does not matter since we can relabel the vertices of K. The only important data is the combinatorial data stored by λ, namely the fact it satisfies the non-singularity condition. Note that the map λ can also be interpreted as a *coloring* of the vertices with elements of $\mathbb{Z}_2^n$ respecting a coloring rule which is the non-singularity condition. Insofar as, vertices being part of the same face must have linearly independent colors.

Now, we denote the set of the characteristic maps over K by $\Lambda(K)$ and consider the $\mathrm{GL}(n, \mathbb{Z}_2)$-action of left multiplication on $\Lambda(K)$, where $\mathrm{GL}(n, \mathbb{Z}_2)$ is the general linear group of degree n over $\mathbb{Z}_2$. One can check that $g \cdot \lambda$ for any $g \in \mathrm{GL}(n, \mathbb{Z}_2)$ still satisfies the non-singularity condition for K. Hence, $\mathrm{GL}(n, \mathbb{Z}_2)$ acts on $\Lambda(K)$ and we will call an orbit of $\mathrm{GL}(n, \mathbb{Z}_2) \curvearrowright \Lambda(K)$ a *DJ class* (of characteristic maps) over K. We denote by $\mathrm{CM}(K) = \mathrm{GL}(n, \mathbb{Z}_2) \setminus \Lambda(K)$.

Since the labels of the vertices of K can be isomorphically modified with an element of a permutation group $\mathfrak{S}_{[m]}$ we can always consider that the face $\{1, \ldots, n\}$ is a maximal face of K. Thus by processing the Gaussian elimination algorithm on an element $\lambda \in \mathrm{CM}(K)$, we can obtain a representative of the DJ class being $\lambda = \left[\begin{array}{c|c} I_n & M \end{array}\right] \in \mathcal{M}(n, m, \mathbb{Z}_2)$, where I_n is the identity matrix of size n, and M is an $n \times (m-n)$ matrix.

We now construct a new object which will store the same combinatorial data as λ as follows. We see $\lambda \in \mathrm{CM}(K)$ as a linear map $\lambda\colon \mathbb{Z}_2^m \to \mathbb{Z}_2^n$ using its matrix representation. Since λ has rank n, by the rank theorem, $\dim(\ker(\lambda)) = m - n$. We take a basis $\{v_1, \ldots, v_{m-n}\}$ of $\ker(\lambda)$. Let us define $\lambda^\star := \left[\begin{array}{c|c|c} v_1 & \cdots & v_{m-n} \end{array}\right] \in \mathcal{M}(m, m-n, \mathbb{Z}_2)$. We have $\lambda\lambda^\star = 0$. The map $\lambda^\star$ is well-defined since $(g \cdot \lambda)\lambda^\star = g \cdot (\lambda\lambda^\star) = 0$ for all $g \in \mathrm{GL}(n, \mathbb{Z}_2)$. We denote by $\bar{\lambda}(1), \ldots, \bar{\lambda}(m)$ the rows of $\lambda^\star$. Then, the corresponding map $\bar{\lambda}\colon [m] \to \mathbb{Z}_2^{m-n}, i \mapsto \bar{\lambda}(i)^t$ is called the *dual characteristic map* associated to λ.

Furthermore, one can see that λ^* is independent from the choice of a basis for the kernel. For all $h \in \mathrm{GL}(m-n, \mathbb{Z}_2)$, $\lambda(\lambda^\star \cdot h) = (\lambda\lambda^\star) \cdot h = 0$. We will then call DJ class of the dual characteristic map $\bar{\lambda}$ the orbit $\overline{\mathrm{CM}}(K) := \{\lambda^\star \cdot h \colon h \in \mathrm{GL}(m-n, \mathbb{Z}_2)\}$.

Remark 5.2.1. There is a one-to-one correspondence between $\mathrm{CM}(K)$ and $\overline{\mathrm{CM}}(K)$. One can notice that if we choose a representative $\lambda = \left[\begin{array}{c|c} I_n & M \end{array}\right]$ (the matrix M is unique for each DJ class) of a DJ class, then we can map this representative to $\bar{\lambda}^t = \left[\begin{array}{c} M \\ \hline I_{m-n} \end{array}\right]$ and we see that

$\lambda\bar{\lambda}^t = \left[\begin{array}{c} M + M \end{array}\right] = 0$. Conversely, from a dual DJ class $\bar{\lambda}$ we can find a representative of the form $\bar{\lambda}^t = \left[\begin{array}{c} M \\ \hline I_{m-n} \end{array}\right]$ and M is unique for each class.

In addition to this correspondence, it was observed in [25, Corollary 7.33] that each DJ class λ and its associated dual $\bar{\lambda}$ share the same combinatorial data and this shows up as a *dual non-singularity condition* for the dual DJ classes.

Proposition 5.2.2. *Let K be a PL sphere on $[m]$ of dimension $n-1$, λ a characteristic map over K, and $\bar{\lambda}$ its dual. Let S be a subset of $[m]$. The following are equivalent:*

1. *$\bar{\lambda}(S^c)$ is a basis of $\mathbb{Z}_2^{m-n}$ (dual non-singularity condition);*
2. *$\lambda(S)$ is a basis of $\mathbb{Z}_2^n$ (non-singularity condition).*

Proof. Let us suppose that $\bar{\lambda}(S^c)$ is a basis of $\mathbb{Z}_2^{m-n}$. We write $\lambda = \left[\begin{array}{c|c} \lambda_{|S} & \lambda_{|S^c} \end{array}\right]$ and $\lambda^\star = \left[\begin{array}{c} \lambda^\star_{|S} \\ \hline \lambda^\star_{|S^c} \end{array}\right]$ which are block matrices representing λ and $\bar{\lambda}$ respectively, we have

$$\lambda\lambda^\star = \lambda_{|S}\lambda^\star_{|S} + \lambda_{|S^c}\lambda^\star_{|S^c} = 0,$$

thus

$$\operatorname{im}\lambda_{|S} \supset \operatorname{im}\lambda_{|S}\lambda^\star_{|S} = \operatorname{im}\lambda_{|S^c}\underbrace{\lambda^\star_{|S^c}}_{\text{is of full rank}} = \operatorname{im}\lambda_{|S^c},$$

and then

$$\operatorname{im}\lambda = \operatorname{im}\lambda_{|S},$$

because we would have a problem with the rank of λ otherwise. Hence, the rank of $\lambda_{|S}$ is n. The converse is proved very similarly by using the co-images instead of the images. $\square$

Remark 5.2.3. One can notice that the proof only requires K to be a simplicial complex. However for the dual characteristic map to be interesting and usable, we will need at least K to be pure, and in our case of study, to be a PL sphere.

Now we know that both a characteristic map and its dual over K store the same combinatorial data. In addition, there is a one-to-one relation between each DJ class and its dual. From this, in all that follows we will not hesitate to mistake the notion of characteristic map with the one of dual characteristic map and use one or another depending on which one is the most convenient for the context.

5.3 The puzzle method for finding characteristic maps over wedged seeds

The goal of this section is to introduce the puzzle method for finding $\mathrm{CM}(K)$, with K a PL sphere, based on [46]. It will be necessary for the next section which will describe a drastic improvement for the mod 2 characteristic maps case. It should be noted that the original method of Choi and Park is applicable to not only mod 2 characteristic maps but also its integral version $\lambda\colon V \to \mathbb{Z}^n$. However, throughout this chapter, we will only deal with the case of mod 2 characteristic maps.

The following definition is an inductive generalization of the wedge operation defined in the introduction.

Definition 5.3.1 ([14])**.** Let K be a simplicial complex on $[m]$, and $J = (j_1, \dots, j_m) \in (\mathbb{Z}_{>0})^m$. We define the wedged simplicial complex $K(J)$ as the simplicial complex obtained after performing $j_v - 1$ wedges on each vertex $v \in [m]$. We define $\mathrm{len}(J) := [\sum_{k=1}^{m} j_k] - m$ to be the total number of wedges performed with J.

Remark 5.3.2. Informally it is the simplicial complex

$$K(J) = \underbrace{\mathrm{wed}_1 \dots \mathrm{wed}_1}_{j_1 - 1 \text{ times}} \dots \underbrace{\mathrm{wed}_v \dots \mathrm{wed}_v}_{j_v - 1 \text{ times}} \dots \underbrace{\mathrm{wed}_m \dots \mathrm{wed}_m}_{j_m - 1 \text{ times}}(L),$$

but writing it in this way is not convenient and requires some work on the notation. Thus, definition 5.3.1 provides a compact notation $K = L(J)$, with L being *the seed of* K (unique up to isomorphism) and J an m-tuple of positive integer representing the wedge operations performed on L to get K. Notice that in this case, the m-tuple J is not unique, in fact the symmetries of L can lead to $L(J)$ being isomorphic to $L(J')$ with $J \neq J'$. More precisely, for a permutation τ on $[m]$ such that $\tau(L) = L$, where $\tau(L)$ is the simplicial complex whose faces σ are the $\tau(f)$ for f a face of L, and for an m-tuple $J = (j_1, \dots, j_m)$, we will have $L(J)$ isomorphic to $L(\tau(J))$, with $\tau(J) = (j_{\tau(1)}, \dots, j_{\tau(m)})$. In addition, one can see that when we write $L(J)$, L does not necessarily need to be a seed. However in all that follows, we will always use this notation when L is the seed of some simplicial complex K.

We can now rephrase our goal. If L is a seed PL sphere on $[m]$ of dimension $n-1$ and $J = (j_1, \dots, j_m)$ is an m-tuple of positive integers, we want to find an algorithm for constructing all DJ classes over the wedged PL sphere $K = L(J)$.

The operation which is converse to the wedge operation is the *link* operation.

Definition 5.3.3 (Link of a face)**.** Let K be a simplicial complex on $[m]$ and $\sigma \in K$ a face of K. Then the Link of σ in K is:

$$\mathrm{lk}_K(\sigma) := \{\tau \in K \mid \sigma \cup \tau \in K, \tau \cap \sigma = \varnothing\}.$$

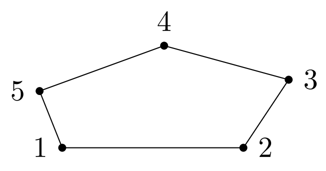
It should be noted that if K is a PL sphere, then the link $\mathrm{lk}_K(\sigma)$ of any face σ of K is also a PL sphere. Furthermore, if K supports a mod 2 characteristic map, so do its links.

Definition 5.3.4. [44] Let λ be a mod 2 characteristic map over K and $\sigma \in K$ a face of K. The *projection of* λ *with respect to* σ is

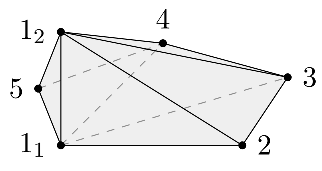
$$(\mathrm{Proj}_\sigma \lambda)(s) = [\lambda(s)] \in \mathbb{Z}_2^n / \langle \lambda(v), v \in \sigma \rangle \simeq \mathbb{Z}_2^{n-|\sigma|},$$

for s a vertex of $\mathrm{lk}_K(\sigma)$. Then, $\mathrm{Proj}_\sigma \lambda$ is well-defined and is in $\mathrm{CM}(\mathrm{lk}_K(\sigma))$. When $\sigma = \{p\}$ is a vertex, we will simply write $\mathrm{Proj}_\sigma \lambda = \mathrm{Proj}_p \lambda$. When $\sigma = \{p, q\}$ is a pair of distinct vertices, we also simply write $\mathrm{Proj}_\sigma \lambda = \mathrm{Proj}_{p,q} \lambda$.

Example 5.3.5. Let us illustrate what we meant by "the operation converse to the wedge is the link operation". Let $\mathcal{P}_5$ be the pentagon on $\{1, \dots, 5\}$. The wedge of $\mathcal{P}_5$ at 1 is illustrated as below.

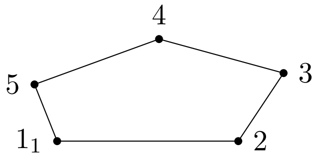


The pentagon $\mathcal{P}_5$

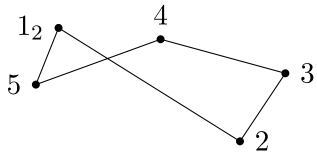


$\mathrm{wed}_1(\mathcal{P}_5)$

The link of $\mathrm{wed}_1(\mathcal{P}_5)$ at vertices 1_1 and 1_2 are two copies of $\mathcal{P}_5$ as follows.



$\mathrm{lk}_{\mathrm{wed}_1(\mathcal{P}_5)}(1_2) \simeq \mathcal{P}_5$



$\mathrm{lk}_{\mathrm{wed}_1(\mathcal{P}_5)}(1_1) \simeq \mathcal{P}_5$

Let $\Lambda = \begin{array}{c} \begin{matrix} 1_1 & 1_2 & 2 & 3 & 4 & 5 \end{matrix} \\ \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \end{array} \in \mathrm{CM}(\mathrm{wed}_1(\mathcal{P}_5))$. The two projections of Λ with respect to the vertices 1_1 and 1_2 are

$$\mathrm{Proj}_{1_1} \Lambda = \begin{array}{c} \begin{matrix} 1_2 & 2 & 3 & 4 & 5 \end{matrix} \\ \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{array}\right] \end{array} \text{ and } \mathrm{Proj}_{1_2} \Lambda = \begin{array}{c} \begin{matrix} 1_1 & 2 & 3 & 4 & 5 \end{matrix} \\ \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{array}\right] \end{array},$$

and they both are characteristic maps over $\mathcal{P}_5$.

We observe that from a pair (K, λ) with K a simplicial complex and $\lambda \in \mathrm{CM}(K)$, we get the pairs $(\mathrm{lk}_K(v), \mathrm{Proj}_v \lambda)$, for v any vertex of K. The operators lk and Proj work together as a pair $(\mathrm{lk}, \mathrm{Proj})$.

The idea of Choi and Park puzzle method relies on what we computed on 5.3.5. If we take a wedged simplicial $K(J)$ there are numerous copies of K in $K(J)$ obtained after some link operations. We simultaneously consider a characteristic map $\Lambda \in \mathrm{CM}(K(J))$, then the consecutive projections of Λ with respect to the same vertex sequence will provide a characteristic map λ over K as follows $(K(J), \Lambda) \overset{(\mathrm{lk},\mathrm{Proj})}{\longrightarrow} \cdots \overset{(\mathrm{lk},\mathrm{Proj})}{\longrightarrow} (K, \lambda)$.

Let us now review the definition of the (pre)diagram $D'(K)$ of K.

Definition 5.3.6 ((Pre)diagram of a PL sphere)**.** Let K be a PL sphere. We define the sets of the vertices $(\mathcal{V})$, edges $(\mathcal{E})$, and realizable squares $(\mathcal{S})$ as follows.

1. $\mathcal{V} = \mathrm{CM}(K)$
2. $\mathcal{E} =$ the set of all characteristic maps in $\mathrm{CM}(\mathrm{wed}_p(K))$ for all vertices p of K.
3. $\mathcal{S} =$ the set of all characteristic maps in $\mathrm{CM}(\mathrm{wed}_{p,q}(K))$ for all distinct unordered pairs of vertices (p, q) of K.

The pair $(\mathcal{V}, \mathcal{E})$ is called the *prediagram* of K and is denoted by $D'(K)$. The triplet $(\mathcal{V}, \mathcal{E}, \mathcal{S})$ is called the *diagram* of K and is denoted by $D(K)$.

Now, here are some important facts concerning the (pre)diagram of a PL sphere.

1. Any edge $\lambda \in \mathcal{E}$ can be associated to the triplet $(\lambda_1, \lambda_2, p)$, where $\lambda_1, \lambda_2 \in \mathcal{V}$ are two projections of λ, namely,

$$\lambda_1 = \mathrm{Proj}_{p_2}(\lambda),\ \lambda_2 = \mathrm{Proj}_{p_1}(\lambda), \text{ and we write } \lambda_1 \overset{p}{\text{------}} \lambda_2 .$$

We thus say that λ_1 and λ_2 are *p-adjacent*. An edge is said to be trivial if $\lambda_1 = \lambda_2$. Every trivial edge (λ, λ, p) is in $\mathcal{E}$.

2. Just like for edges, any realizable square $\lambda \in \mathcal{S}$ can be associated to the following projections onto K.

$$\begin{aligned} \lambda_{1,1} &= \mathrm{Proj}_{p_2,q_2}(\lambda), \\ \lambda_{1,2} &= \mathrm{Proj}_{p_2,q_1}(\lambda), \\ \lambda_{2,1} &= \mathrm{Proj}_{p_1,q_2}(\lambda), \\ \lambda_{2,2} &= \mathrm{Proj}_{p_1,q_1}(\lambda), \end{aligned} \qquad \text{and we write} \qquad \begin{tikzcd} \lambda_{1,1} \arrow[r, dash, "p"] \arrow[d, dash, "q"'] & \lambda_{2,1} \arrow[d, dash, "q"] \\ \lambda_{1,2} \arrow[r, dash, "p"'] & \lambda_{2,2} \end{tikzcd} ,$$

   with $\lambda_{i,j} \in \mathcal{V}$ for $i, j = 1, 2$, and each edge being in $\mathcal{E}$.

The diagram $D(K)$ stores the "rules" which are required for completing the puzzle. But now, we need a "board" in order to play this puzzle game. Let us remind that the main objective is to create $\mathrm{CM}(L(J))$ from $\mathrm{CM}(L)$. The board should intuitively represent the adjacent relations between copies of L inside $L(J)$, as in 5.3.5.

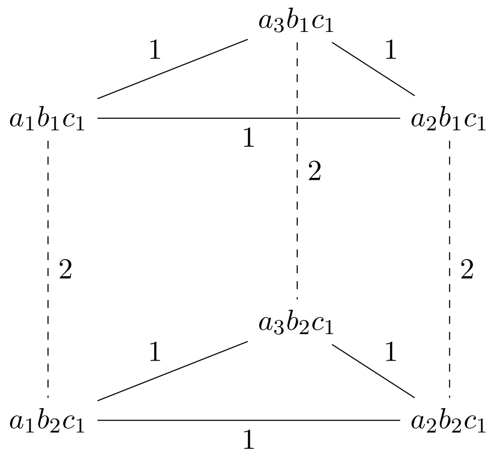
Let $J = (j_1, \ldots, j_m)$ be an m-tuple of positive integers. We define the graph $G(J)$ of J as the 1-skeleton of the simplicial complex $\Delta^J := \Delta^{j_1-1} \times \cdots \times \Delta^{j_m-1}$. Each edge of $G(J)$ can be written uniquely as

$$e = v_1 \times v_2 \times \ldots \times v_{p-1} \times e_p \times v_{p+1} \times \ldots \times v_m,$$

where v_i is a vertex of $\Delta^{j_i-1}, i = 1, \ldots, m, i \neq p$ and e_p is an edge of Δ^{j_p-1}. We endow $G(J)$ with an edge coloring where each e is colored with the corresponding $p \in [m]$.

Remark 5.3.7. A subsquare of $G(J)$ is a subgraph of $G(J)$ that comes from a 2-face of Δ^J which has 2 edges. We can see that the opposed edges of a square have pairwise the same color.

Example 5.3.8. Put $J = (3, 2, 1)$. The graph $G(J)$ is the 1-skeleton of $\Delta^2 \times \Delta^1 \times \Delta^0$ we label its nodes as (a_i, b_j, c_k) with $i = 1, 2, 3$, $j = 1, 2$, and $k = 1$. One can notice that since we did not perform any wedge operation on the vertex 3, there is no edges colored as 3. We represent it as follows.



A *realizable puzzle* is a coloring of the nodes of $G(J)$ by DJ classes of K by respecting the edges and squares coloring rules given by the diagram of K. Namely, it is a map $\pi\colon \mathrm{Vert}(G(J)) \to \mathcal{V} = \mathrm{CM}(K)$, which is in addition an edge coloring preserving pseudo-graph homomorphism $\pi\colon G(J) \to D'(K)$ such that each image of a subsquare of $G(J)$ is a realizable square of $D(K)$. The set of realizable puzzles will be denoted as $\mathcal{RP}(K, J)$.

We thus have the following theorem.

Theorem 5.3.9 ([46], Theorem 5.4)**.** *For any PL sphere K, there is a one-to-one correspondence*

$$\mathrm{CM}(K(J)) \overset{1:1}{\longleftrightarrow} \mathcal{RP}(K, J).$$

Remark 5.3.10. In this theorem, we consider the set of realizable puzzles of K with wedge J. To compute it, we need to compute the set of characteristic maps over K by the Garrison and Scott algorithm. We thus need K to be of minimal dimension for the Garrison and Scott algorithm to finish quickly (see 5.5), namely K should be a seed.

The puzzle method gives us directly the following algorithm for finding $\mathrm{CM}(L(J))$, for L a seed. Its main idea is to enumerate all possible puzzles, namely put every possible combination of stone positions on the board, and then check if some combinations make the puzzle realizable.

Algorithm 5.3.11 (The old puzzle algorithm)**.**

— **Input:** L a seed PL sphere on $[m]$ with $\dim(L) = n-1$, $J \in \mathbb{Z}^m_{>0}$ and $\mathrm{CM}(L)$ the set of the DJ classes over L.

— **Output:** The set $\mathcal{RP}(L, J)$ of all realizable puzzles $\pi : G(J) \to D'(L)$.

— **Initialization:**

$D(L) \leftarrow$ diagram of L.

$\pi \leftarrow$ The empty map.

$\mathcal{RP}(L, J) \leftarrow \varnothing$.

$\mathcal{PT}(L, J) \leftarrow \varnothing$. (Puzzle Trash, used for a memoization purpose)

— **Procedure:**

1. For each node v of the graph $G(J)$, set $\pi(v) = \lambda$ for some $\lambda \in \mathrm{CM}(L)$. The puzzle π should not be in $\mathcal{PT}(L, J)$.
2. If the image of some edge of $G(J)$ by π is not in $D(L)$ then add π to $\mathcal{PT}(L, J)$ and go to (1).
3. If the image of some subsquare of $G(J)$ by π is not realizable then add π to $\mathcal{PT}(L, J)$ and go to (1).
4. Add π to $\mathcal{RP}(L, J)$.

One can notice that the problem of this puzzle algorithm is that it needs to check if all the edges and the subsquares of the puzzle are in the diagram (Procedure (2) and(3)), and this for each possible puzzle. That requires a lot of computation time (see 5.5.1 for a detailed computation of the complexity of this algorithm), and we thus need a drastic improvement.

In the following section, we will create a constructive and procedural algorithm for the puzzle method to become applicable and faster.

5.4 The improved puzzle algorithm

5.4.1 Computation of the edge rules

Let us understand more explicitly what an edge of the prediagram represents.

First of all, let us introduce some notations. For $\lambda \in \mathrm{CM}(K)$ and $k = 1, \dots, n$, we may assume that $\lambda(k) = e_k$, the kth vector of the canonical basis, and we will use the following notations.

$$\lambda = \begin{array}{cc} & \begin{array}{ccccc|ccccc} 1 & \cdots & k & \cdots & n & n+1 & \cdots & n+j & \cdots & m \end{array} \\ \begin{array}{c} 1 \\ \\ k \\ \\ n \end{array} & \left[\begin{array}{ccccc|ccccc} 1 & 0 & \cdots & \cdots & 0 & \lambda(n+1)_1 & \cdots & \lambda(n+j)_1 & \cdots & \lambda(m)_1 \\ 0 & \ddots & \ddots & & \vdots & \vdots & & \vdots & & \vdots \\ \vdots & \ddots & 1 & \ddots & \vdots & \lambda(n+1)_k & \cdots & \lambda(n+j)_k & \cdots & \lambda(m)_k \\ \vdots & & \ddots & \ddots & 0 & \vdots & & \vdots & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & \lambda(n+1)_n & \cdots & \lambda(n+j)_n & \cdots & \lambda(m)_n \end{array}\right] \end{array}$$

$$= \begin{array}{cc} & \begin{array}{ccccc|ccccc} 1 & \cdots & k & \cdots & n & n+1 & \cdots & n+j & \cdots & m \end{array} \\ \begin{array}{c} 1 \\ \\ k \\ \\ n \end{array} & \left[\begin{array}{ccccc|ccccc} 1 & 0 & \cdots & \cdots & 0 & \bar{\lambda}(1)_1 & \cdots & \bar{\lambda}(1)_j & \cdots & \bar{\lambda}(1)_{m-n} \\ 0 & \ddots & \ddots & & \vdots & \vdots & & \vdots & & \vdots \\ \vdots & \ddots & 1 & \ddots & \vdots & \bar{\lambda}(k)_1 & \cdots & \bar{\lambda}(k)_j & \cdots & \bar{\lambda}(k)_{m-n} \\ \vdots & & \ddots & \ddots & 0 & \vdots & & \vdots & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & \bar{\lambda}(n)_1 & \cdots & \bar{\lambda}(n)_j & \cdots & \bar{\lambda}(n)_{m-n} \end{array}\right] \end{array},$$

for $j = 1, \dots, m-n$. Note that it coincides with the dual characteristic map notation. Let $\lambda \in \mathrm{CM}(\mathrm{wed}_p(K))$, and let us denote its two projections by $\lambda_1 = \left[\begin{array}{c|c} & \bar{\lambda}_1(1) \\ I_n & \vdots \\ & \bar{\lambda}_1(n) \end{array}\right]$ and $\lambda_2 = \left[\begin{array}{c|c} & \bar{\lambda}_2(1) \\ I_n & \vdots \\ & \bar{\lambda}_2(n) \end{array}\right]$. They are p-adjacent by definition.

On one hand, in the case $p \in \{1, \dots, n\}$, the characteristic map λ which makes λ_1 and λ_2 p-adjacent can be written in the following form, called the basic form of λ (defined in [46]).

$$\lambda = \begin{array}{cc} & \begin{array}{cccccccc|c} 1 & 2 & & p_1 & p_2 & & n-1 & n & n+1, \dots, m \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \\ p_1 \\ p_2 \\ \\ n-1 \\ n \end{array} & \left[\begin{array}{cccccccc|c} 1 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & \bar{\lambda}_1(1) = \bar{\lambda}_2(1) \\ 0 & 1 & \ddots & \vdots & \vdots & & & \vdots & \bar{\lambda}_1(2) = \bar{\lambda}_2(2) \\ \vdots & \ddots & \ddots & 0 & \vdots & & & \vdots & \vdots \\ \vdots & & \ddots & 1 & 0 & & & \vdots & \bar{\lambda}_1(p) \\ \vdots & & & 0 & 1 & \ddots & & \vdots & \bar{\lambda}_2(p) \\ \vdots & & & \vdots & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & & & \vdots & \vdots & \ddots & 1 & 0 & \bar{\lambda}_1(n-1) = \bar{\lambda}_2(n-1) \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 1 & \bar{\lambda}_1(n) = \bar{\lambda}_2(n) \end{array}\right] \end{array},$$

if we denote by $\phi := \bar{\lambda}_1(p) + \bar{\lambda}_2(p)$ the row vector corresponding to the change at row p between $\bar{\lambda}_1$ and $\bar{\lambda}_2$. The edge in the pre-diagram can be seen as follows.

$$\lambda_1 = \left[\begin{array}{c|c} & \bar{\lambda}_1(1) \\ & \vdots \\ I_n & \bar{\lambda}_1(p) \\ & \vdots \\ & \bar{\lambda}_1(n) \end{array}\right] \xrightarrow{\phi,p} \lambda_2 = \left[\begin{array}{c|c} & \bar{\lambda}_1(1) \\ & \vdots \\ I_n & \bar{\lambda}_1(p)+\phi \\ & \vdots \\ & \bar{\lambda}_1(n) \end{array}\right]. \tag{5.4.1}$$

Conversely, one can see that $\lambda_2 \xrightarrow{\phi,p} \lambda_1$ as well.

On the other hand, if $p > n$, we can find $1 \leqslant k \leqslant n$ such that the kth coordinate $\lambda_1(p)_k$ is equal to 1. We can then reduce $\lambda_1(p)$ onto e_k by performing the operations $\ell_l \leftarrow \ell_l + \ell_k \pmod 2$, with ℓ_i the ith line of λ_1 for $l \neq k$. After this, $\lambda_1(k)$ becomes $\lambda_1(p)$. If we write $a_j = \bar{\lambda}_1(k)_j = \lambda_1(n+j)_k$, $j = 1, \dots, m-n$, we have

$$\lambda_1 = \begin{array}{cc} & \begin{array}{ccccccc|c|c} 1 & 2 & \cdots & k & \cdots & n-1 & n & p & (n+j) \neq p \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \\ k \\ \\ n-1 \\ n \end{array} & \left[\begin{array}{ccccccc|c|c} 1 & 0 & \cdots & \lambda_1(p)_1 & \cdots & \cdots & 0 & 0 & \bar{\lambda}_1(1)_j + \lambda_1(p)_1 a_j \\ 0 & 1 & \ddots & \vdots & & & \vdots & \vdots & \bar{\lambda}_1(2)_j + \lambda_1(p)_2 a_j \\ \vdots & \ddots & \ddots & \vdots & & & \vdots & 0 & \vdots \\ \vdots & & \ddots & 1 = \lambda_1(p)_k & & & \vdots & 1 & a_j \\ \vdots & & & \vdots & \ddots & \ddots & \vdots & 0 & \vdots \\ \vdots & & & \vdots & \ddots & 1 & 0 & \vdots & \bar{\lambda}_1(n-1)_j + \lambda_1(p)_{n-1} a_j \\ 0 & \cdots & \cdots & \lambda_1(p)_n & \cdots & 0 & 1 & 0 & \bar{\lambda}_1(n)_j + \lambda_1(p)_n a_j \end{array}\right] \end{array}.$$

If we switch the columns k and p, we get the same as previously after wedging

$$\lambda = \begin{array}{cc} & \begin{array}{cccccccc|c|c} 1 & 2 & & p_1 & p_2 & & n-1 & n & k & (n+j) \neq p \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \\ k_1 \\ k_2 \\ \\ n-1 \\ n \end{array} & \left[\begin{array}{cccccccc|c|c} 1 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & \lambda_1(p)_1 & \bar{\lambda}_1(1)_j + \lambda_1(p)_1 a_j \\ 0 & 1 & \ddots & \vdots & \vdots & & & \vdots & \vdots & \bar{\lambda}_1(2)_j + \lambda_1(p)_2 a_j \\ \vdots & \ddots & \ddots & 0 & \vdots & & & \vdots & \vdots & \vdots \\ \vdots & & \ddots & 1 & 0 & & & \vdots & 1 & a_j \\ \vdots & & & 0 & 1 & \ddots & & \vdots & b_0 & b_j \\ \vdots & & & \vdots & 0 & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & & & \vdots & \vdots & \ddots & 1 & 0 & \vdots & \bar{\lambda}_1(n-1)_j + \lambda_1(p)_{n-1} a_j \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 1 & \lambda_1(p)_n & \bar{\lambda}_1(n)_j + \lambda_1(p)_n a_j \end{array}\right] \end{array},$$

with $b_0 = 1$ because otherwise $\{1, \dots, k, \dots, n\}$ is not a face of $\mathrm{wed}_p(K)$ anymore. We then perform the projection on the column p_1 and switch columns p and k again, we obtain

$$\lambda_2 = \begin{array}{c} \\ 1 \\ 2 \\ \\ k \\ \\ n-1 \\ n \end{array} \begin{array}{c} \begin{array}{cccccccc|c|c} 1 & 2 & \cdots & k & \cdots & n-1 & n & p & & (n+j)\neq p \end{array} \\ \left[\begin{array}{ccccccc|c|c} 1 & 0 & \cdots & \lambda_1(p)_1 & \cdots & \cdots & 0 & 0 & \bar{\lambda}_1(1)_j + \lambda_1(p)_1 a_j \\ 0 & 1 & \ddots & \vdots & & & \vdots & \vdots & \bar{\lambda}_1(2)_j + \lambda_1(p)_2 a_j \\ \vdots & \ddots & \ddots & \vdots & & & \vdots & 0 & \vdots \\ \vdots & & \ddots & 1 = \lambda_1(p)_k & & & \vdots & 1 & b_j \\ \vdots & & & \vdots & \ddots & \ddots & \vdots & 0 & \vdots \\ \vdots & & & \vdots & \ddots & 1 & 0 & \vdots & \bar{\lambda}_1(n-1)_j + \lambda_1(p)_{n-1} a_j \\ 0 & \cdots & \cdots & \lambda_1(p)_n & \cdots & 0 & 1 & 0 & \bar{\lambda}_1(n)_j + \lambda_1(p)_n a_j \end{array}\right] \end{array},$$

and so if we reduce again to have $\lambda_2(k)$ being equal to e_k we get

$$\lambda_2 = \begin{array}{c} \\ 1 \\ 2 \\ \\ k \\ \\ n-1 \\ n \end{array} \begin{array}{c} \begin{array}{cccccccc|c} 1 & 2 & \cdots & k & \cdots & n-1 & n & p & (n+j)\neq p \end{array} \\ \left[\begin{array}{ccccccc|c|c} 1 & 0 & \cdots & 0 & \cdots & \cdots & 0 & \lambda_1(p)_1 & \bar{\lambda}_1(1)_j + \lambda_1(p)_1 (a_j+b_j) \\ 0 & 1 & \ddots & \vdots & & & \vdots & \vdots & \bar{\lambda}_1(2)_j + \lambda_1(p)_2 (a_j+b_j) \\ \vdots & \ddots & \ddots & \vdots & & & \vdots & \vdots & \vdots \\ \vdots & & \ddots & 1 & & & \vdots & 1 & b_j \\ \vdots & & & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & & & \vdots & \ddots & 1 & 0 & \vdots & \bar{\lambda}_1(n-1)_j + \lambda_1(p)_{n-1} (a_j+b_j) \\ 0 & \cdots & \cdots & 0 & \cdots & 0 & 1 & \lambda_1(p)_n & \bar{\lambda}_1(n)_j + \lambda_1(p)_n (a_j+b_j) \end{array}\right] \end{array}.$$

Thus, the row vector ψ corresponding to this edge has cordinates $\psi_j = a_j + b_j$, for $j = 1, \dots, m-n$ and $(n+j) \neq p$, and $\psi_p = 0$ for $(n+j) = p$, the edge in the pre-diagram can be seen as follows

$$\lambda_1 = \left[\begin{array}{c|c} & \bar{\lambda}_1(1) \\ I_n & \vdots \\ & \bar{\lambda}_1(n) \end{array}\right] \xrightarrow{\psi,p} \lambda_2 = \left[\begin{array}{c|c} & \bar{\lambda}_1(1) + \lambda_1(p)_1\psi \\ I_n & \vdots \\ & \bar{\lambda}_1(n) + \lambda_1(p)_n\psi \end{array}\right], \tag{5.4.2}$$

and $\lambda_2 \xrightarrow{\psi,p} \lambda_1$.

Thus, if two DCM (dual characteristic maps) $\bar{\lambda}_1$ and $\bar{\lambda}_2$ are p-adjacent, the implying edge in the diagram can be associated to a unique (since in our calculations we chose a unique representative for each dual DJ class) row vector ϕ in (5.4.1) or ψ in (5.4.2).

Conversely, if we take $\lambda_1, \lambda_2 \in \mathrm{CM}(K)$ such that there exists a relation of the form (5.4.1) or (5.4.2) according to a vertex p, then processing the same calculations in the opposite direction allows us to build a characteristic map λ over $\mathrm{wed}_p(K)$ with $\{\lambda(p_1), \lambda(p_2)\}$ being a unimodular set. We now have the following proposition in [44].

Proposition 5.4.1 ([44, Proposition 4.4]). *Let K be a PL sphere with vertex set V. For $v \in V$ and let λ be a characteristic map on $\mathrm{wed}_v(K)$ such that $\{\lambda(v_1), \lambda(v_2)\}$ is a unimodular set. Then λ is non-singular if and only if $\mathrm{Proj}_{v_1}(\lambda)$ and $\mathrm{Proj}_{v_2}(\lambda)$ are non-singular. Furthermore, λ is uniquely determined by $\mathrm{Proj}_{v_1}(\lambda)$ and $\mathrm{Proj}_{v_2}(\lambda)$.*

Since λ_1 and λ_2 are chosen non-singular, then λ is non-singular. We write $\lambda := \lambda_1 \wedge_v \lambda_2$ and we call it the *v-wedge* of λ_1 and λ_2.

It then provides us the following lemma concerning the edges of the diagram.

Lemma 5.4.2. *Let K be a PL sphere on $[m]$ with $\dim(K) = n-1$, $\lambda_1, \lambda_2 \in \mathrm{CM}(K)$ and $p \in [m]$. Then, the following are equivalent*

1. *There exists $\lambda \in \mathrm{CM}(\mathrm{wed}_p(K))$ such that $\lambda = \lambda_1 \wedge_p \lambda_2$;*
2. *There exists a $\mathbb{Z}_2$ row vector ψ of size $(m-n)$ such that $\bar{\lambda}_2 = h\bar{\lambda}_1$*

$$\text{with } h = \left[\begin{array}{c|c} & \lambda_1(p)_1\psi \\ & \vdots \\ I_n & \lambda_1(p)_i\psi \\ & \vdots \\ & \lambda_1(p)_n\psi \\ \hline 0 & I_{m-n} \end{array}\right] \in \mathcal{M}(m,m,\mathbb{Z}_2), \text{ and } \bar{\lambda}_i = \left[\begin{array}{c} \bar{\lambda}_i(1) \\ \vdots \\ \bar{\lambda}_i(n) \\ \hline I_{m-n} \end{array}\right], \text{ for } i = 1, 2.$$

Proof. The proof comes directly when one notices that the transformations (5.4.1) (since $\lambda_1(p) = e_p, p = 1, \dots, n$) and (5.4.2) can be seen as follows.

$$\left[\begin{array}{c} \bar{\lambda}_1(1) + \lambda_1(p)_1\psi \\ \vdots \\ \bar{\lambda}_1(i) + \lambda_1(p)_i\psi \\ \vdots \\ \bar{\lambda}_1(n) + \lambda_1(p)_n\psi \\ \hline I_{m-n} \end{array}\right] = \left[\begin{array}{c|c} & \lambda_1(p)_1\psi \\ & \vdots \\ I_n & \lambda_1(p)_i\psi \\ & \vdots \\ & \lambda_1(p)_n\psi \\ \hline 0 & I_{m-n} \end{array}\right] \left[\begin{array}{c} \bar{\lambda}_1(1) \\ \vdots \\ \bar{\lambda}_1(i) \\ \vdots \\ \bar{\lambda}_1(n) \\ \hline I_{m-n} \end{array}\right] \Leftrightarrow \bar{\lambda}_2 = h\bar{\lambda}_1,$$

with h of the desired form. □

Remark 5.4.3. This construction can be clarified as follows. Two dual DJ classes $\bar{\lambda}_1$ and $\bar{\lambda}_2$ are p-adjacent if and only if we can go from $\bar{\lambda}_1$ to $\bar{\lambda}_2$ by adding the same row vector ψ to each row of $\bar{\lambda}_1$ at coordinate j which satisfies $\lambda_1(p)_j = 1$. One can then see that from any pair of characteristic maps over K, it is very easy to check if they are p-adjacent or not, for some vertex p of K, using linear algebra. Furthermore, this construction gives us a very direct way of computing the prediagram of K simply from $\mathrm{CM}(K)$.

For all that follows, since the matrices h depend on p, λ_1 and ψ, we will denote such h by a triplet (p, λ_1, ψ).

For a dual characteristic map $\bar{\lambda}_0 \in \overline{\mathrm{CM}}(K)$, let us write $H_p(\bar{\lambda}_0)$ the set of the matrices $h = (p, \lambda_0, \psi)$ from all the edges $\bar{\lambda}_0 \overset{p}{\text{———}} h\bar{\lambda}_0$ which are in the prediagram. From now on, we will write such an edge $\bar{\lambda}_0 \overset{p,h}{\text{———}} h\bar{\lambda}_0$.

Remark 5.4.4. Here are some important properties of $H_p(\bar{\lambda}_0)$.

1. It is easy to check that this set is actually a commutative subgroup of $GL(m, \mathbb{Z}_2)$ whose elements act from the left on the set of the $\bar{\lambda} \in \overline{\mathrm{CM}}(K)$ which are p-adjacent to $\bar{\lambda}_0$.
2. Furthermore, one can notice that if $q \neq p$ is another vertex of K, and if $\lambda_0(p) = \lambda_0(q)$, then if $h = (p, \lambda_0, \psi) \in H_p(\bar{\lambda}_0)$ we have $h' = (q, \lambda_0, \psi) \in H_q(\bar{\lambda}_0)$ so $\psi(p) = \psi(q) = 0$ which means that elements of $H_p(\bar{\lambda}_0)$ cannot modify the color of the vertices having the same color as $\lambda_0(p)$ and since the roles of p and q can be switched, $H_q(\bar{\lambda}_0) = H_p(\bar{\lambda}_0)$.

3. The action of $H_p(\bar{\lambda}_0)$ on the p-adjacent component of $\bar{\lambda}_0$ is free (and transitive) by construction. Therefore if $\bar{\lambda}_1$ and $\bar{\lambda}_2$ are p-adjacent, then $H_p(\bar{\lambda}_1) = H_p(\bar{\lambda}_2)$.

Choi and Park found that a realizable puzzle is automatically entirely determined if one chooses the image of a node v of $G(J)$ and of every nodes w connected to v by an edge as in the following proposition.

Proposition 5.4.5 ([46, Proposition 4.3])**.** *Let v be any fixed node of $G(J)$ and suppose that there are two realizable puzzles $\pi, \pi' : G(J) \to D'(K)$. Then $\pi = \pi'$ if and only if $\pi(v) = \pi'(v)$ and $\pi(w) = \pi'(w)$ for every node w of $G(J)$ which is adjacent to v.*

The latter proposition is the cornerstone of the new algorithm for finding realizable puzzles in a constructive way. In fact, when we have a seed PL sphere L on $[m]$ and an m-tuple J, we just need the image by a realizable puzzle π of few nodes of $G(J)$ to get the entire DJ class corresponding to π.

5.4.2 Computation of the square rules

Let us recall that, from Theorem 5.4 in [46], we know a puzzle is realizable if and only if all of its subsquares are realizable. So let us find a condition for the squares to be realizable. More precisely, since we want to describe a constructive algorithm, we consider the smallest step we can make in the construction of the realizable puzzle. This step is completing a square which would have one of its nodes "missing".

An *incomplete pq*-square is a square of the form

$$\begin{tikzcd}[arrows=-] \bar{\lambda} \arrow[r, "{p,\alpha}"] \arrow[d, "{q,\beta}"'] & \bar{\lambda}^{\alpha} \arrow[d, "{q,?}"] \\ \bar{\lambda}^{\beta} \arrow[r, "{p,?}"'] & ? \end{tikzcd} \text{ , with } p \neq q.$$

Completing this square requires to

— Find an element $\bar{\lambda}^{\gamma}$ such that $\bar{\lambda}^{\alpha} \overset{q}{\text{———}} \bar{\lambda}^{\gamma}$ and $\bar{\lambda}^{\beta} \overset{p}{\text{———}} \bar{\lambda}^{\gamma}$ are in the prediagram of K. This can be easily done with a computer algorithm by 5.4.2. Such element will be called *a possible missing piece* of the square.

— Check if a possible missing piece $\bar{\lambda}^{\gamma}$ makes the square $\begin{tikzcd}[arrows=-] \bar{\lambda} \arrow[r, "p"] \arrow[d, "q"'] & \bar{\lambda}^{\alpha} \arrow[d, "q"] \\ \bar{\lambda}^{\beta} \arrow[r, "p"'] & \bar{\lambda}^{\gamma} \end{tikzcd}$ a realizable square.

We will call such $\bar{\lambda}^{\gamma}$ *<u>the</u> missing piece* of the square.

The missing piece of a square is unique, due to 5.4.5 applied on the J corresponding to wed_{pq}, which makes the terminology correct.

We will prove the following lemma.

Lemma 5.4.6. *For an incomplete pq-square as before we have these three possible cases*

1. *If there exists no possible missing piece, then the square cannot be completed;*

2. *If there exists more than one possible missing pieces for the square, then we can explicitly compute and find the missing piece of the puzzle among them;*
3. *If there exists only a single possible missing piece for the square, then it is the missing piece of this square and we can once more compute it explicitly.*

Proof. Statement (1) is immediate by the definitions we used.

For proving (2), we consider that there exists $\bar{\lambda}^\gamma$ and $\bar{\lambda}^{\gamma\prime}$ two possible missing pieces of this square. We have

$$\begin{tikzcd}
\bar{\lambda} \arrow[r, dash, "{p, \alpha_p}"] \arrow[d, dash, "{q, \beta_q}"'] & \bar{\lambda}^\alpha \arrow[d, dash, "{q, \alpha_q}"'] \arrow[ddr, dash, bend left, "{q, \alpha'_q}"] & \\
\bar{\lambda}^\beta \arrow[r, dash, "{p, \beta_p}"] \arrow[drr, dash, bend right, "{p, \beta'_p}"'] & \bar{\lambda}^\gamma \arrow[dr, dash, dashed, "{q, \tilde{\alpha}}", "{p, \tilde{\beta}}"'] & \\
& & \bar{\lambda}^{\gamma\prime}
\end{tikzcd},$$

with $\alpha_p \in H_p(\bar{\lambda}), \beta_q \in H_q(\bar{\lambda})$ and $\alpha_q, \alpha'_q \in H_q(\bar{\lambda}^\alpha), \beta_p, \beta'_p \in H_p(\bar{\lambda}^\beta)$ and the dashed line being found as follows

— $\bar{\lambda}^{\gamma\prime} = \alpha'_q \bar{\lambda}^\alpha = \alpha'_q(\alpha_q \bar{\lambda}^\gamma) = \tilde{\alpha}\bar{\lambda}^\gamma$, so the edge $\bar{\lambda}^\gamma \overset{q, \tilde{\alpha}}{\rule{3em}{0.4pt}} \bar{\lambda}^{\gamma\prime}$ is in the prediagram of K since $\tilde{\alpha} \in H_q(\lambda^\gamma)$;

— Similarly, $\bar{\lambda}^\gamma \overset{p, \tilde{\beta}}{\rule{3em}{0.4pt}} \bar{\lambda}^{\gamma\prime}$ is in the prediagram of K since $\tilde{\beta} = \beta'_p \beta_p \in H_p(\lambda^\gamma)$.

We write $\tilde{\alpha} = (q, \lambda^\gamma, \phi)$ and $\tilde{\beta} = (p, \lambda^\gamma, \psi)$.

Since by 5.4.1 a generator between two dual DJ classes of K is unique, we have $\tilde{\alpha} = \tilde{\beta}$ and the equality between such objects is given by $\phi = \psi$ and the color of p, respectively q, in the p-adjacent, respectively q-adjacent component of $\bar{\lambda}^\gamma$ remains unchanged (by 5.4.4 (2)). Thus since $\bar{\lambda}$ is in both of these connected components, this leads to $\lambda(p) = \lambda(q)$.

Hence the only case there would be several possible missing pieces is when $\lambda(p) = \lambda(q)$. In that case, we can easily find which dual DJ class is the missing piece (i.e. creates a realizable square). It is exactly given by $\bar{\lambda}^\gamma := \alpha_p \beta_q \bar{\lambda}$. The corresponding square is

$$\begin{tikzcd}
\bar{\lambda} \arrow[r, dash, "{p, \alpha_p}"] \arrow[d, dash, "{q, \beta_q}"'] & \bar{\lambda}^\alpha \arrow[d, dash, "{q, \beta_q}"] \\
\bar{\lambda}^\beta \arrow[r, dash, "{p, \alpha_p}"'] & \bar{\lambda}^\gamma
\end{tikzcd}.$$

Let us show that it is a realizable square. We write $\alpha_p = (p, \lambda^\gamma, \phi)$ and $\beta_q = (q, \lambda^\gamma, \psi)$. Without loss of generality, we can suppose that $p = 1$ and $q = n + 1$. We have

$$\bar{\lambda} = \begin{bmatrix} \bar{\lambda}(1) \\ \bar{\lambda}(\star) \\ I_{m-n} \end{bmatrix}, \bar{\lambda}^{\alpha} = \begin{bmatrix} \bar{\lambda}(1)+\phi \\ \bar{\lambda}(\star) \\ I_{m-n} \end{bmatrix}, \bar{\lambda}^{\beta} = \begin{bmatrix} \bar{\lambda}(1)+\psi \\ \bar{\lambda}(\star) \\ I_{m-n} \end{bmatrix}, \text{ and } \bar{\lambda}^{\gamma} = \begin{bmatrix} \bar{\lambda}(1)+\psi+\phi \\ \bar{\lambda}(\star) \\ I_{m-n} \end{bmatrix}.$$

Thus

$$\lambda \wedge_1 \lambda^{\alpha} = \begin{array}{c} \begin{array}{ccccc} 1_1 & 1_2 & 2,\dots,n & q & q+1,\dots,m \end{array} \\ \left[\begin{array}{cc|c|c|c} 1 & 0 & 0 & 1 & \bar{\lambda}(1)+\phi \\ 0 & 1 & 0 & 1 & \bar{\lambda}(1) \\ \mathbf{0} & \mathbf{0} & I_{n-2} & \mathbf{0} & \bar{\lambda}(\star) \end{array}\right] \end{array},$$

and

$$\lambda^{\beta} \wedge_1 \lambda^{\gamma} = \begin{array}{c} \begin{array}{ccccc} 1_1 & 1_2 & 2,\dots,n & q & q+1,\dots,m \end{array} \\ \left[\begin{array}{cc|c|c|c} 1 & 0 & 0 & 1 & \bar{\lambda}(1)+\psi+\phi \\ 0 & 1 & 0 & 1 & \bar{\lambda}(1)+\psi \\ \mathbf{0} & \mathbf{0} & I_{n-2} & \mathbf{0} & \bar{\lambda}(\star) \end{array}\right] \end{array}.$$

By 5.4.2, the edge $\lambda \wedge_1 \lambda^{\alpha} \xrightarrow{q,\Delta} \lambda^{\beta} \wedge_1 \lambda^{\gamma}$ where $\Delta = (q, \lambda^{\beta} \wedge_1 \lambda^{\gamma}, \psi)$ is in the prediagram of $\mathrm{wed}_p(K)$ and thus represents a DJ class of $\mathrm{wed}_q(\mathrm{wed}_p(K)) = \mathrm{wed}_{p,q}(K)$ implying the square is realizable and proving (2).

Now, we need to prove (3). So let us consider that there exists only a unique possible missing piece. We have $\lambda(p) \neq \lambda(q)$ otherwise we can just look at the above calculations and easily compute the missing piece. We then have two cases.

Firstly, if p and q are together in a maximal face of K. By relabeling and without loss of generality we can consider that $p = 1$ and $q = 2$. We consider the following $(1,2)$-square

$$\begin{tikzcd} \bar{\lambda} \arrow[r, dash, "{1,\alpha_1}"] \arrow[d, dash, "{2,\beta_2}"'] & \bar{\lambda}^{\alpha} \arrow[d, dash, "{2,\beta_2}"] \\ \bar{\lambda}^{\beta} \arrow[r, dash, "{1,\alpha_1}"'] & \bar{\lambda}^{\gamma} \end{tikzcd}.$$

Let us show that it is realizable. We write $\alpha_1 = (1, \bar{\lambda}, \phi)$ and $\beta_2 = (2, \bar{\lambda}, \psi)$, we can then describe the dual DJ classes

$$\bar{\lambda} = \begin{bmatrix} \bar{\lambda}(1) \\ \bar{\lambda}(2) \\ \bar{\lambda}(\star) \\ I_{m-n} \end{bmatrix}, \bar{\lambda}^{\alpha} = \begin{bmatrix} \bar{\lambda}(1)+\phi \\ \bar{\lambda}(2) \\ \bar{\lambda}(\star) \\ I_{m-n} \end{bmatrix}, \bar{\lambda}^{\beta} = \begin{bmatrix} \bar{\lambda}(1) \\ \bar{\lambda}(2)+\psi \\ \bar{\lambda}(\star) \\ I_{m-n} \end{bmatrix}, \text{ and } \bar{\lambda}^{\gamma} = \begin{bmatrix} \bar{\lambda}(1)+\phi \\ \bar{\lambda}(2)+\psi \\ \bar{\lambda}(\star) \\ I_{m-n} \end{bmatrix}.$$

The 1-wedges of this square are as follows

$$\lambda \wedge_1 \lambda^\alpha = \begin{array}{c} \begin{array}{ccccc} 1_1 & 1_2 & 2 & 3,\dots,n & n+1,\dots,m \end{array} \\ \left[\begin{array}{cc|c|c|c} 1 & 0 & 0 & \mathbf{0} & \bar\lambda(1)+\phi \\ 0 & 1 & 0 & \mathbf{0} & \bar\lambda(1) \\ 0 & 0 & 1 & \mathbf{0} & \bar\lambda(2) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{n-3} & \bar\lambda(\star) \end{array}\right] \end{array},$$

and

$$\lambda^\beta \wedge_1 \lambda^\gamma = \begin{array}{c} \begin{array}{ccccc} 1_1 & 1_2 & 2 & 3,\dots,n & n+1,\dots,m \end{array} \\ \left[\begin{array}{cc|c|c|c} 1 & 0 & 0 & \mathbf{0} & \bar\lambda(1)+\phi \\ 0 & 1 & 0 & \mathbf{0} & \bar\lambda(1) \\ 0 & 0 & 1 & \mathbf{0} & \bar\lambda(2)+\psi \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{n-3} & \bar\lambda(\star) \end{array}\right] \end{array},$$

Again, 5.4.2 gives that the edge $\lambda \wedge_1 \lambda^\alpha \xrightarrow{\quad 2,\Delta \quad} \lambda^\beta \wedge_1 \lambda^\gamma$, with $\Delta = (2, \lambda \wedge_1 \lambda^\alpha, \psi)$, is in the prediagram of $\mathrm{wed}_1(K)$ and thus represents a DJ class of $\mathrm{wed}_2(\mathrm{wed}_1(K)) = \mathrm{wed}_{1,2}(K)$, namely, the square is realizable.

Secondly, if p and q appear in none of the maximal faces. By relabeling, we can suppose that $p = 1$ and $q = n+1$. Hence $\lambda(p) = e_1$ and $\lambda(q)$ is an arbitrary vector different from e_1 (We again forget this case since we have computed it before). We consider the following square.

$$\begin{tikzcd} \bar\lambda \arrow[r, dash, "{1,(1,\lambda,\phi)}"] \arrow[d, dash, "{n+1,(n+1,\lambda,\psi)}"'] & \bar\lambda^\alpha \arrow[d, dash, "{n+1,(n+1,\lambda^\alpha,\delta)}"] \\ \bar\lambda^\beta \arrow[r, dash, "{1,(1,\lambda^\beta,\epsilon)}"'] & \bar\lambda^\gamma \end{tikzcd} .$$

Recall that $\psi_1 = \delta_1 = 0$ (see (5.4.2)), so inside this square, the only coordinate of $\lambda(n+1)$ which can change is its first coordinate. Since $\lambda(n+1) \notin \{\mathbf{0}, e_1\}$, we must have an index $k > 1$ such that $\lambda(n+1)_k = 1$ (of course $n > 1$), for simplifying the notations, we suppose that $k = 2$, then $(n+1, \lambda, \psi)$ will modify at least $\bar\lambda(k)$.We first get

$$\bar\lambda^\alpha = \begin{bmatrix} \bar\lambda(1)+\phi \\ \bar\lambda(k) \\ \bar\lambda(\star) \\ I_{m-n} \end{bmatrix} \text{ and } \bar\lambda^\beta = \begin{bmatrix} \bar\lambda(1)+\lambda(n+1)_1 \times \psi \\ \bar\lambda(k)+\psi \\ \bar\lambda(\star)+\lambda(n+1)_\star \times \psi \\ I_{m-n} \end{bmatrix}.$$

From the two remaining edges, we obtain two forms for $\bar\lambda^\gamma$ as follows.

$$\bar\lambda^\gamma = \begin{bmatrix} \bar\lambda(1)+\phi+[\lambda(n+1)_1+\phi_1] \times \delta \\ \bar\lambda(k)+\delta \\ \bar\lambda(\star)+\lambda(n+1)_\star \times \delta \\ I_{m-n} \end{bmatrix} \text{ and } \bar\lambda^\gamma = \begin{bmatrix} \bar\lambda(1)+\lambda(n+1)_1 \times \psi+\epsilon \\ \bar\lambda(k)+\psi \\ \bar\lambda(\star)+\lambda(n+1)_\star \times \psi \\ I_{m-n} \end{bmatrix}.$$

Hence we get from the kth row that $\delta = \psi$. The first row provides the following explicit formula $\epsilon = \lambda(n+1)_1 \times \psi + \phi + [\lambda(n+1)_1 + \phi_1] \times \psi = \phi + \phi_1 \times \psi$, but will not be used here. The first 1-wedge of the square is

$$\lambda \wedge_1 \lambda^\alpha = \begin{array}{c} \begin{array}{cc|c|c|c} 1_1 & 1_2 & 2,\dots,n & n+1 & n+j = n+2,\dots,m \end{array} \\ \left[\begin{array}{cc|c|c|c} 1 & 0 & 0 & \lambda(n+1)_1 + \phi_1 & \bar\lambda(1)_j + \phi_j \\ 0 & 1 & 0 & \lambda(n+1)_1 & \bar\lambda(1)_j \\ \mathbf{0} & \mathbf{0} & I_{n-2} & \lambda(n+1)_\star & \bar\lambda(\star)_j \end{array}\right] \end{array}.$$

The first coordinate of $\lambda^\gamma(n+1)$ is

$$\begin{aligned} \lambda^\gamma(n+1)_1 = \bar\lambda^\gamma(1)_1 &= [\bar\lambda(1) + \phi + (\lambda(n+1)_1 + \phi_1) \times \psi]_1 \\ &= \bar\lambda(1)_1 + \phi_1 + (\lambda(n+1)_1 + \phi_1) \times \underbrace{\psi_1}_{=0} \\ &= \lambda(n+1)_1 + \phi_1. \end{aligned}$$

Furthermore the other coordinates $\lambda^\gamma(n+1)_i$, for $i = 2, \dots, n$, are

$$\begin{aligned} \lambda^\gamma(n+1)_i = \bar\lambda^\gamma(i)_1 &= [\bar\lambda(i) + (\lambda(n+1)_1 + \phi_1) \times \psi]_1 \\ &= \bar\lambda(i)_1 + (\lambda(n+1)_1 + \phi_1) \times \underbrace{\psi_1}_{=0} \\ &= \lambda(n+1)_i. \end{aligned}$$

From this we obtain the second 1-wedge as follows

$$\lambda^\beta \wedge_1 \lambda^\gamma = \begin{array}{c} \begin{array}{cc|c|c|c} 1_1 & 1_2 & 2,\dots,n & n+1 & n+j = n+2,\dots,m \end{array} \\ \left[\begin{array}{cc|c|c|c} 1 & 0 & 0 & \lambda(n+1)_1 + \phi_1 & \bar\lambda(1)_j + \phi_j + (\lambda(n+1)_1 + \phi_1) \times \psi_j \\ 0 & 1 & 0 & \lambda(n+1)_1 & \bar\lambda(1)_j + \lambda(n+1)_1 \times \psi_j \\ \mathbf{0} & \mathbf{0} & I_{n-2} & \lambda(n+1)_\star & \bar\lambda(\star)_j + \lambda(n+1)_\star \times \psi_j \end{array}\right] \end{array}.$$

One can see by 5.4.2 that the edge $\lambda \wedge_1 \lambda^\alpha \xrightarrow{n+1,\Delta} \lambda^\beta \wedge_1 \lambda^\gamma$, with $\Delta = (n+1, \lambda \wedge_1 \lambda^\alpha, \psi)$, is in the prediagram of $\mathrm{wed}_1(K)$ and thus provides a DJ class over $\mathrm{wed}_{1,n+1}(K)$ as desired. As a conclusion, in every cases we found that if there exists a possible missing piece for an incomplete pq-square, then it leads to the unique missing piece of the square proving (3). $\square$

Remark 5.4.7. This proof actually gives us the explicit and only form that realizable squares can take. Thus, while 5.4.2 provides the set of edges $\mathcal{E}$ of the diagram of K, 5.4.6 gives us the set of realizable squares $\mathcal{S}$ of $D(K)$, namely all the DJ classes over K after two consecutive wedges. More precisely, it gives us a map from the set of incomplete pq-squares onto $\mathrm{CM}(\mathrm{wed}_{p,q}(K)) \cup \{\varnothing\}$ which gives either the missing piece of an incomplete pq-square if it can be completed or $\varnothing$ otherwise.

As desired, 5.4.2 and 5.4.6 provides us all the data of the diagram of K. So we have now the basic steps of an algorithm which will enumerate all the realizable puzzles over a wedged seed PL sphere $L(J)$.

5.4.3 Description of the new puzzle algorithm

Let us recall that a node v in the graph $G(J)$ can be represented by an m-tuple

$$(v_1, v_2, \dots, v_m),$$

with $1 \leqslant v_1 \leqslant j_1, 1 \leqslant v_2 \leqslant j_2, \dots, 1 \leqslant v_m \leqslant j_m$. We define the depth of a node v in the graph $D(J)$ as $d(v) = |\{v_i \neq 1 | i = 1, \dots, m\}| \leqslant m$. For example we have $d((1, \dots, 1)) = 0$. In addition, we order the nodes of $G(J)$ having the same depth by lexicographic order. The depth d together with the lexicographic order gives a total order $\preceq$ on the nodes of $G(J)$.

We will define by $D'(L)_p$ the prediagram of L which is restricted to the edges colored with p. We will also denote by $D'(L)_p(\lambda)$ the connected component of λ in $D'(L)_p$. Since the action of $H_p(\bar{\lambda})$ is free and transitive on a connected component, then $D'(L)_p(\lambda)$ is a complete graph.

We then have the following theorem.

Theorem 5.4.8. *Given a wedged seed PL sphere $L(J)$ and its set of DJ classes $\mathrm{CM}(L)$, there exists an algorithm for finding the DJ classes set $\mathrm{CM}(L(J))$ whose complexity only depends on the size of $\mathrm{CM}(L)$, the length of J, and the Picard number of L. The algorithm is recursive and is described in 5.4.9.*

Algorithm 5.4.9.

— **Input:** L a seed PL sphere on $[m]$ with $\dim(L) = n - 1$, $J \in \mathbb{Z}^m_{>0}$ and $\mathrm{CM}(L)$ the set of the DJ classes over L.

— **Output:** The set $\mathcal{RP}(L, J)$ of all realizable puzzles $\pi : G(J) \to D(L)$.

— **Initialization:**

$D'(L) \leftarrow$ Prediagram of L using 5.4.2.
$\pi \leftarrow$ The empty map.
$\mathcal{RP}(L, J) \leftarrow \varnothing$.
$v \leftarrow (1, \dots, 1)$.
$d \leftarrow 0$.

— **Procedure**(d, v, π)**:**

1. If $d > d(J)$ add π to $\mathcal{RP}(L, J)$;
2. Else
   (a) $\Lambda \leftarrow \mathrm{CM}(L)$
   (b) For every node $\nu \prec v$, if there is an edge $\nu \overset{p}{\text{———}} v$ in $G(J)$ then
      i. $\Lambda \leftarrow \Lambda \cap D'(L)_p(\pi(\nu))$.
      ii. For every node $\nu \prec \mu \prec v$, if there is an edge $\mu \overset{q}{\text{———}} v$ in $G(J)$.

      — Find the unique node u completing the square $\begin{tikzcd} u \arrow[r, dash, "p"] \arrow[d, dash, "q"'] & \mu \arrow[d, dash, "q"] \\ \nu \arrow[r, dash, "p"'] & v \end{tikzcd}$, with $u \prec \nu$ and $d(u) < d(v)$.

      — $(\star)$ $\Lambda = \Lambda \cap$ the missing piece of $\begin{tikzcd} \pi(u) \arrow[r, dash, "p"] \arrow[d, dash, "q"'] & \pi(\mu) \arrow[d, dash, "q"] \\ \pi(\nu) \arrow[r, dash, "p"'] & ? \end{tikzcd}$ by 5.4.6.

(c) If Λ is empty then terminate the current call.

(d) Else, for every $\lambda \in \Lambda$:

i. Copy π onto some new puzzle π'.

ii. $\pi'(v) \leftarrow \lambda$.

iii. If v is the biggest element of depth d then set v' to be the smallest element of depth $d+1$ and call **Procedure**$(d+1, v', \pi')$.

iv. Else, set v' to be the the next element in lexicographical order of depth d and call **Procedure**(d, v', π').

Proof of the algorithm. We denote by $G(J)_{\prec v}$ the graph $G(J)$ restricted to the nodes smaller than v and fix the loop invariant to be $A(v)$=*"At the beginning of the call, the image by the puzzle π of every node $\nu \prec v$ is defined, and every edges or subsquares of* $G(J)_{\prec v}$ *is sent by π to an edge of the prediagram or to a realizable square, respectively"*. The algorithm has two states.

The first one is when we still deal with nodes v of depth smaller or equal to 1 and in this case there is no subsquares in $G(J)_{\prec v}$. At the end of this state, the image or every edges between $(1, \dots, 1)$ and v is guaranteed to be in the prediagram of L by condition $(2, b, ii)$.

The second one is when $d(v) \geqslant 2$. The initialization of the loop invariant for this state is trivial from the first state of the algorithm and since there is no squares in $G(J)_{\prec v_0}$, for v_0 the smallest element of depth 2. Let us suppose that for some $v \succ v_0$, $A(v)$ is verified. We achieve one more step in the loop and we check that $A(v')$ holds for v' the element coming after v for $\prec$. Since $A(v)$ is verified, the only edges and subsquares of $G(J)_{\prec v'}$ we need to consider are the ones containing v. At the end of (2,b), there are two possible cases:

— The set Λ is empty and thus the current call is terminated in (2,c) since we either did not find any missing piece for some such subsquares or found two distinct missing pieces for some pair of subsquares,

— The set Λ is not empty and this means that the image by π of all such edges are in the prediagram of L, from Procedure (2,b,i), and that the image by π of all the considered subsquares provide the same missing piece, from Procedure (2,b,ii), and are thus realizable.

At the end of the loop if the call keeps running, it means that $A(v')$ is verified, as desired.

A call adds a puzzle π to the result set if and only if it reaches the node v_∞ of highest depth and the greatest for the lexicographic order according to procedure (1). Then, this means that every nodes of $G(J)$ is colored by the puzzle π and all subsquares of $G(J)_{\preceq v_\infty} = G(J)$ are realizable, which means that π is realizable, as desired. $\square$

5.5 Complexity comparisons

Let us compare the performances of the three methods we have to obtain every characteristic maps over a wedged seed PL sphere $L(J)$.

1. The Garrison and Scott algorithm directly applied on $L(J)$;
2. 5.3.11 applied on (L, J);
3. 5.4.9 applied on (L, J).

In all that follows, the elementary operations we consider for the algorithm are the summation of $\mathbb{Z}_2$-vectors (this operation is done in $\mathcal{O}(1)$ since we can use a binary encoding for such vectors), and the access to an element of an array.

We will denote by $\Pi(J) = \prod_{k=1}^{m} j_k$ the number of vertices of $G(J)$. Also recall that $\text{len}(J) = \sum_{k=1}^{m} j_k - m$ represents the number of wedge operations performed with J.

First of all, let us show that 5.4.9 is way better than 5.3.11.

5.5.1 Puzzle algorithm, old versus new

Let us first describe more explicitly the complexity of 5.3.11.

This algorithm computes the following.

— In the initialization, we compute all the DJ classes of L, $\mathrm{wed}_p(L), p \in [m]$ and $\mathrm{wed}_{p,q}(L)$, $p, q \in [m], p \neq q$ with the Garrison and Scott algorithm.

— For each of the $\Pi(J)^{|\,\mathrm{CM}(L)|}$ possible puzzles,

  — In procedure (2), we check the

$$\sum_{k=1}^{m} \binom{j_k}{2} \prod_{l \neq k} j_l = \sum_{k=1}^{m} \frac{j_k(j_k-1)}{2} \prod_{l \neq k} j_l = \frac{1}{2}\Pi(J)\,\mathrm{len}(J)$$

  edges of the puzzle.

  — In procedure (3), we check the

$$\sum_{1 \leqslant k < l \leqslant m} \binom{j_k}{2}\binom{j_l}{2} \prod_{i \neq l,k} j_i = \frac{1}{4}\Pi(J) \sum_{1 \leqslant k < l \leqslant m} (j_k-1)(j_l-1)$$

  squares of the puzzle.

Then it leads to the following proposition.

Proposition 5.5.1. *The complexity of 5.3.11 is*

$$\mathcal{O}\left(m^2 \cdot GS(\mathrm{wed}_{p,q}(L)) + \Pi(J)^{|\,\mathrm{CM}(L)|+1} \cdot \left(\mathrm{len}(J) + \sum_{1 \leqslant k < l \leqslant m} (j_k-1)(j_l-1)\right)\right),$$

with $GS(K)$ being the complexity of the Garrison and Scott algorithm on a given simplicial complex K. Notice that we omitted the term $m \cdot GS(\mathrm{wed}_p(L))$ since its complexity is $o(m^2 \cdot GS(\mathrm{wed}_{p,q}(L)))$.

On the contrary, 5.4.9 computes the following.

— To provide the input data, we only need to compute $\mathrm{CM}(L)$.

— Let us give an upper bound for the number of possible realizable puzzles. For each node v of the graph $G(J)$, we will associate an image $\pi(v)$ which is determined from its neighbours of smaller or equal depth. However 5.4.5 stays that once the nodes of depth $\leqslant 1$ have their images determined, then the puzzle is either entirely determined or will possess some square which is not realizable and thus will not be realizable. As a consequence, for depth > 1, the algorithm will simply verify that all the squares are indeed realizable for the remaining nodes. We call the depth $\leqslant 1$ coloring of the graph $G(J)$ a *determining corner* of the graph. Thus, for finding an upper bound on the number of realizable puzzles, we only need to count the number of determining corners. The algorithm will associate an image λ^0 to the node $(1, \ldots, 1)$ among the $|\,\mathrm{CM}(L)|$ choices possible. Then the algorithm will choose one DJ class p-adjacent to λ^0 for every nodes w which are p-adjacent to v (of depth 1, and for every p) and there are $(\mathrm{len}(J) - 1)$ such w. After looking at 5.4.1, one can notice that for any DJ class λ^0, there is a maximum of $2^{\mathrm{Pic}(L)}$ DJ classes which can be p-adjacent to λ^0 since two p-adjacent edges differ by a $\psi \in \mathbb{Z}_2^{\mathrm{Pic}(L)}$. Thus, the number of realizable puzzles is bounded by $|\,\mathrm{CM}(L)| \cdot (2^{\mathrm{Pic}(L)})^{\mathrm{len}(J)}$.

— For each remaining node ν, we will check if its image by π can be the missing piece of a square. To do so, we check the number of preceding pair of edges ν is part of. The number of such edges is bounded by $\operatorname{len}(J)$ and we take two of them. Thus this whole process has a complexity bounded by $(|\operatorname{Vert}(G(J))|)^{\operatorname{len}(J)^2} = \Pi(J)^{\operatorname{len}(J)^2}$. This is checked for every determining corners.

This thus leads to the following proposition.

Proposition 5.5.2. *The complexity of 5.4.9 is*

$$\mathcal{O}\left(\Pi(J)^{\operatorname{len}(J)^2} \cdot |\operatorname{CM}(L)| \cdot 2^{\operatorname{Pic}(L)\operatorname{len}(J)}\right).$$

Remark 5.5.3. In practice, the number of DJ classes over L tends to be very small when the dimension increases for a fixed Picard number. Also notice that we provided a very rough upper bound on the number of characteristic maps which are p-adjacent for some vertex p. In fact, the structure of the prediagram of L influences a lot on the time complexity. Namely, many cases of determining corners will be discarded in no time. In addition, when the components of J increase all together, this gives more restrictions when computing the depth 2 and the number of determining corners providing a realizable square will then drastically decrease.

One can see that when $\operatorname{len}(J)$ becomes bigger, then the complexity of 5.3.11 becomes drastically worse than the one of 5.4.9.

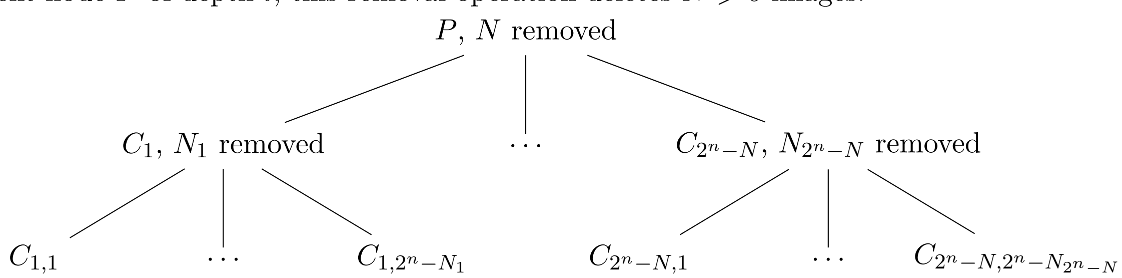
5.5.2 The performance of Algorithm 5.4.9 versus a direct use of the Garrison and Scott algorithm

Let J be an m-tuple of positive integers with $j := \operatorname{len}(J)$. A DJ class of $L(J)$ (the ones which are searched with the GS algorithm) is written as

$$\lambda = \begin{array}{c} \begin{array}{ccccc} 1\dots n+j & n+j+1 & n+j+2 & \dots & m+j \end{array} \\ \left[\begin{array}{c|cccc} I_{n+j} & \lambda_{n+j+1} & \lambda_{n+j+2} & \dots & \lambda_{m+j} \end{array}\right] \end{array}.$$

In order to apply 5.4.9 we need two steps. The first one is to compute $\operatorname{CM}(L)$ by the Garrison and Scott algorithm on the seed L. If $j > 0$, since this algorithm is applied on L, it will for sure be more efficient than the Garrison and Scott algorithm on $L(J)$. The second step is the main part of the algorithm and depends only on j, on p the Picard number of L, and on the number of elements in $\operatorname{CM}(L)$.

Let us compute the minimal complexity for the Garrison and Scott algorithm applied on a pure simplicial complex K on $[m]$, with $\dim(K) = n-1$. First, recall that the main idea of the Garrison and Scott algorithm is to progressively enumerate all the possible images $\lambda(n+l) \in \mathbb{Z}_2^n$ for $l = 1, \dots, m-n$ by removing the sums of the form $\sum_{i\in\sigma}\lambda(i)$ with $\sigma \cup \{n+l\} \in K \cap [n+l]$ from the set $\mathbb{Z}_2^n$. The branch-and-bound tree has a depth equal to $m-n$. Let us say that, for a parent node P of depth l, this removal operation deletes $N \geqslant 0$ images.



Since removing one element is an elementary operation, this makes the complexity to be $\mathcal{O}(N)$ for this step. The node P will have $2^n - N$ children, say $C_1, \ldots, C_{2^n-N}$. For each child C_k, we will have to remove the sums of the form $\sum_{i\in\sigma} \lambda(i)$, with $\sigma \cup \{n+l+1\} \in L\cap[n+l+1]$, say N_k distinct sums from $\mathbb{Z}_2^n$. Thus, increasing the number of child increases the time complexity exponentially.

So the thinner the tree is at step l, the smaller the time complexity will be. The smallest tree for the Garrison and Scott algorithm has its nodes having only one son at each step, except for the last one (multiplicity equal to $|\operatorname{CM}(L(J))|$). The depth of the tree will of course be equal to $p = m - n$ and the number of operations done for each node is the number of colors we remove so it is 2^n. The total complexity is then equal to $\mathcal{O}(p \cdot 2^n)$.

As an example, let us compute the explicit amount of sums $\sum_{i\in\sigma} \lambda(i)$ such that $\sigma \cup \{n+1\} \in L \cap [n+1]$, namely, the first step in the Garrison and Scott algorithm. We consider K a PL sphere. It should satisfy the *pseudo manifold* condition which is that any $(n-2)$-face should be included in exactly two $(n-1)$-faces. Let us suppose that the $(n-2)$-face $f = \{1, \ldots, n-1\} \subset F_0 = \{1, \ldots, n\} (\in K)$ is also included in $\{1, \ldots, n-1, n+1\}$ this means that every subset of f is a face of K and this gives us 2^{n-1} sums which are removed from the 2^n first available choices for $\lambda(n+1)$, since $\lambda(i) = e_i$, $i = 1, \ldots, n$. Of course some other sums may be removed by the other faces (of dimension $\leqslant n-1$) which contain $n+1$.

This computation also assures us that since a $(n-2)$-face f containing the vertex $n+l$, $l = 1\ldots, p$ must be in some $(n-1)$-face F, then this face will contain at least $n-p$ vertices from the face F_0, which leads to removing at least 2^{n-p} sums at each step. Once more, the Picard number p is fixed and this increases exponentially with respect to the dimension $n-1$.

Now if we come back to our actual case, we study the complexity of the Garrison and Scott algorithm for $K = L(J)$ having dimension $n + j - 1$. Thus the lower bound for the complexity of the Garrison and Scott algorithm on $L(J)$ is $\mathcal{O}\left(p \cdot 2^{n+j}\right)$.

This latter complexity depends exponentially on n. Then for a fixed number of wedge operations j and Picard number p, if we increase n, then at some point the puzzle algorithm will be faster than the Garrison and Scott algorithm.

Here is a benchmark obtained at Picard number 4 in which we can observe this result with $|J| = 3$.

(n, m)	(2,6)	(3,7)	(4,8)	(5,9)	(6,10)	(7,11)
Number of PL spheres	1	4	21	142	733	1190
Garrison and Scott (time for one)	0.05s	0.85s	0.78s	1.93s	6s	112s
5.4.9 (time for one)	0.35s	0.31s	0.11s	0.02s	0.013s	0.010s

Table 5.1 – Comparison between the Garrison and Scott algorithm and 5.4.9.

Note that for obtaining the previous table, we used a variation of the Garrison and Scott algorithm which in practice makes the computation of $\operatorname{CM}(L)$ faster when L is a seed and the Picard number p is small. A description of this algorithm can be found in 5.7.

5.6 Conclusion and discussion

The main improvement of 5.4.9 is that the computation of mod 2 characteristic maps on $L(J)$ with a fixed Picard number p now only depends on

— The complexity of the Garrison and Scott algorithm on the seed L which is of smaller dimension and has fewer vertices, and on

— The number of wedge operations j performed on L and the size of $\mathrm{CM}(L)$.

This algorithm can also determine quite quickly if a determining corner does not lead to a realizable puzzle (if a square does not have a missing piece or if two missing pieces coming from two different squares provide a different coloring for the same node then the algorithm stops completing this puzzle).

For toric topological purposes, one would like to extend this algorithm for $\mathbb{Z}$ characteristic maps. However, it is known that missing pieces for a square are not unique in this case (see [46]) and this refrains us from using the same algorithm. However when one works on a complex toric variety X, it is one-to-one associated with a fan which in some cases is associated to a characteristic map λ on a simplicial complex K. Then, the real projection $X^{\mathbb{R}}$ of X will lead to the (mod 2) fan which is in the same cases associated to the (mod 2) reduction $\lambda^{\mathbb{R}}$ of λ over the same K. So $X^{\mathbb{R}}$ will support (at least) one characteristic map $\lambda^{\mathbb{R}}$. Conversely, if K supports a (mod 2) characteristic map $\lambda^{\mathbb{R}}$ then this simplicial complex is a good candidate for being associated with fan and then to a complex toric variety. Furthermore, if K does not support any (mod 2) characteristic map then it cannot be associated to a complex toric variety. Hence, it is enough to consider the seeds L which support (mod 2) characteristic maps and use the puzzle algorithm to construct the small covers associated with them. In fact, if a seed does not support any (mod 2) characteristic map, then its wedges will also not support any as a consequence of the puzzle method.

For the case of wedge of polygons in [47], the authors show that determining corners can lead for sure to a realizable puzzle by providing some additional rules on the determining corners. This shows that with some modification on 5.4.9 we can stop it at depth 1. One could then wonder if this is true for a general case and try to find some rules for more general cases.

5.7 The IDCM Garrison and Scott algorithm (Appendix)

One interesting fact about 5.4.9 is that it only requires to compute the Garrison and Scott algorithm on L which is a seed PL sphere. Seed PL spheres which are not direct product of simplices have their dual characteristic maps $\bar{\lambda}$ which all are *injective*, namely, $\bar{\lambda}(i) \neq \bar{\lambda}(j), i \neq j$. This fact can be used for creating a variation of the Garrison and Scott algorithm.

This section describes a different version of the Garrison and Scott algorithm. Since this algorithm is computed on IDCM (Injective Dual Characteristic Maps), the colors for the vertices of L will be in $\mathbb{Z}_2^p$, so one can "intuitively" think that for a "small" Picard number and an n "big enough", this algorithm will be faster than the Garrison and Scott algorithm. In this section we only give some rough idea of the complexities without proper proofs. Only a the benchmark table at the end allows to convince ourselves that our intuition seems correct.

Let L be a seed PL sphere on $[m]$ with $\dim(L) = n - 1$.

As said before, any characteristic map on L will have its dual injective, so we can use the following variation of the Garrison and Scott algorithm for finding $\mathrm{CM}(L)$, we will call it the IDCM Garrison and Scott algorithm.

Algorithm 5.7.1 (A modification of Algorithm 4.1 in [75])**.**

- **Input:** CF = the union of the power sets of all cofacets of L.
- **Output:** Γ = the list of $\mathbb{Z}_2$ vectors $(\lambda_1, \ldots, \lambda_m)$ such that the last $m-n$ vectors form the standard basis $\mathbb{Z}_2^{m-n}$.
- **Initialization:**

  $\lambda_{n+1} \leftarrow (1, 0, \ldots, 0), \lambda_{n+2} \leftarrow (0, 1, \cdots, 0), \ldots, \lambda_m \leftarrow (0, 0, \ldots, 1)$.
  $\Gamma \leftarrow \varnothing$
  $S \leftarrow$ the list of nonzero elements of $\mathbb{Z}_2^{m-n}$
  $i \leftarrow n$
- **Procedure:**
  1. Set $S_i \leftarrow S \setminus \{\lambda_j, j > i\}$.
  2. For all $I \in CF$ of the form $I = \{i\} \cup \{i_1, \ldots, i_k\}$ with $1 \leqslant i \leqslant i_1 \leqslant \cdots \leqslant i_k$, remove the vector $\lambda_{i_1} + \cdots + \lambda_{i_k}$ from the list S_i.
  3. If $i = n$, then STOP.
  4. If $S_i = \varnothing$, then $i \leftarrow i + 1$ and go to (3).
  5. Set $\lambda_i \leftarrow$ the 1st element of S_i and remove it from S_i
  6. If $i = 1$, then add $(\lambda_1, \ldots, \lambda_m)$ to Γ and go to (3).
  7. Set $i \leftarrow i - 1$ and go to (1).

Let us explain why this algorithm is intuitively more efficient than the classic GS algorithm for small Picard numbers.

For a simplicial complex K on $[m]$, and $i \in [m]$, we define $K_{\leqslant i} := \{f \cup i \subset [i] \mid f \in K\}$, the set of faces of K having all their vertices in the set $[i]$. The *star* of K at a face $F \in K$ is denoted as $\mathrm{st}_K(F) := \{G \in K \mid F \subset G\}$, and is the set of all faces of K containing the face F. We denote by $\mathcal{O}(K)$ the PL sphere whose maximal faces are the complementary of the maximal faces of K, we then have $\dim(\mathcal{O}(K)) = \mathrm{Pic}(K) - 1$.

The parameters which impact the complexity of the branch-and-bound algorithm are the following:

- The size of the research tree;
- The size of the sets we use for deleting branches of the tree.

— The complexity of the basic operations we use (here we deal with sum of vectors in $\mathbb{Z}_2^a$ for some a).

On one hand, the GS algorithm will work on the complete tree $\mathcal{T}_{\mathrm{GS}}$ whose nodes are every possible vectors $\lambda_{n+k} \in \mathbb{Z}_2^n, k = 1, \ldots, \mathrm{Pic}(L)$ such that

$$\begin{array}{cccc} 1 \ldots n & n+1 & \ldots & m \\ \end{array}$$
$$\left[\begin{array}{c|ccc} I_{n+j} & \lambda_{n+1} & \ldots & \lambda_{n+i} \end{array}\right]$$

is a characteristic map of $L_{\leqslant i}$. We have $|\mathcal{T}_{GS}| = (2^n - 1)^{\mathrm{Pic}(L)}$. The removal of a branch of the tree is made by finding the elements of $\mathrm{st}_{L_{\leqslant i}}(i)$ for $i = n+1, \ldots, m$ in L to make the wanted characteristic map respect the non-singularity condition. If we suppose that we calculated $\mathrm{st}_{L_{\leqslant i}}(i)$ for $i = 1, \ldots, \mathrm{Pic}(L)$ at the very beginning, we then have to compute the $|\,\mathrm{st}_{L_{\leqslant i}}(i)| \geqslant 2^{(n-1)} - (n-1)$ or $= 0$ linear combinations of vectors in $\mathbb{Z}_2^n$ since there is either at least one maximal face F containing i and then each proper sub-face of F leads to one linear combination or no such maximal faces (but in this case less branches are deleted).

On the other hand, the IDCM Garrison and Scott algorithm works on a tree which is of depth n and the number of children starts from $2^{\mathrm{Pic}(L)} - 1$ and is decremented at each step. The tree $\mathcal{T}_{\mathrm{IDCM\ GS}}$ is then of size $|\mathcal{T}_{\mathrm{IDCM\ GS}}| = \frac{(2^{\mathrm{Pic}(L)}-1-\mathrm{Pic}(L))!}{(2^{\mathrm{Pic}(L)}-1-\mathrm{Pic}(L)-n)!}$. Moreover, the removal of branches is done by finding the faces in $co(L)_{\geqslant i}$ and in this case $|co(L)_{\geqslant i}| \geqslant 2^{(\mathrm{Pic}(L)-1)} - (\mathrm{Pic}(L) - 1)$. Furthermore the linear combinations are computed in $\mathbb{Z}_2^{\mathrm{Pic}(L)}$.

Table 5.2 gives approximations of the value associated to these complexities for Picard number 4.

(n,m)	(2,6)	(3,7)	(4,8)	(5,9)	(6,10)	(7,11)	(8,12)	(9,13)	(10,14)	(11,15)
$\lvert\mathcal{T}_{\mathrm{GS}}\rvert$	81	2e3	5e4	9e5	2e7	3e8	4e9	7e10	1e12	1e13
$\lvert\,\mathrm{st}_{L_{\leqslant i}}(i)\rvert \geqslant$	2	5	12	27	58	121	248	503	1e3	2e3
$\lvert\mathcal{T}_{\mathrm{IDCM\ GS}}\rvert$	110	990	8e+3	6e4	3e5	2e6	7e6	2e7	4e7	4e7
$\lvert co(L)_{\geqslant i}\rvert \geqslant$	11	11	11	11	11	11	11	11	11	11

Table 5.2 – Comparative table of the complexities.

For Picard number 4 and $n > 4$, the tree is smaller with the IDCM version of the algorithm and there are fewer linear combinations to compute, which also are done in a smaller vector space. However we cannot tell precisely about the number of branches which will be removed. We just know that the branch removal operations are faster with the IDCM version.

Table 5.3 gives some benchmark on the time efficiency of the IDCM algorithm if compared to the Garrison and Scott version.

* *
*

(n, m)	(2,6)	(3,7)	(4,8)	(5,9)	(6,10)	(7,11)
Number of PL spheres	1	4	21	142	733	1190
G-S (time for all)	0.2ms	3ms	50ms	1.5s	25s	353s
IDCM G-S (time for all)	0.4ms	7ms	82ms	1.7s	14s	15s

Table 5.3 – Time efficiency comparison between the Garrison and Scott algorithm and the IDCM modification.

6 Toric wedge induction and the toric lifting problem

> *"If we have no idea why a statement is true, we can still prove it by induction."*
>
> *Gian-Carlo Rota*

This chapter is based on a joint work with Suyoung Choi and Hyeontae Jang, published in the Journal of the London Mathematical Society [41]. We leverage the database of toric-colorable seeds with Picard number 4 obtained in Chapter 4 and solve the toric lifting problem in that case, using the combinatorics of some specific binary matroid together with a framework that we name "toric wedge induction". The merit for this framework goes to Choi and Park who first used it for proving the projectivity of some toric manifolds [45].

6.1 Introduction

Let K be a simplicial complex on the set $[m] = \{1, \ldots, m\}$. We define the *polyhedral product* $(\underline{X}, \underline{Y})^K$ of K with respect to a pair (X, Y) of topological spaces as follows:

$$(\underline{X}, \underline{Y})^K := \bigcup_{\sigma \in K} \{(x_1, \ldots, x_m) \in X^m \mid x_i \in Y \text{ when } i \notin \sigma\}.$$

Here, D^d represents the d-dimensional disk, defined as $D^d = \{\mathbf{x} \in \mathbb{R}^d \mid \|\mathbf{x}\|_2 \leqslant 1\}$, and S^{d-1} denotes its boundary sphere of dimension $d-1$. The *moment-angle complex* $\mathcal{Z}_K$ of K is then defined as $(\underline{D^2}, \underline{S^1})^K$, and the *real moment-angle complex* $\mathbb{R}\mathcal{Z}_K$ of K is $(\underline{D^1}, \underline{S^0})^K$. We observe that the T^1-action on the pair (D^2, S^1) leads to the canonical action of the m-dimensional torus $T^m = (S^1)^m$ on $\mathcal{Z}_K$. Additionally, there is an S^0-action on the pair (D^1, S^0). For clarity and consistency in our terminology throughout this chapter, we treat S^0 as the additive group $\mathbb{Z}_2 = \mathbb{Z}_2\mathbb{Z}$ with two elements $\{0, 1\}$. This, then, yields the canonical $\mathbb{Z}_2^m$-action on $\mathbb{R}\mathcal{Z}_K$.

It is noteworthy that when an r-dimensional subtorus H of T^m acts freely on $\mathcal{Z}_K$, the resulting quotient space $\mathcal{Z}_K/H$ admits a well-behaved torus action $T^m/H \cong T^{m-r}$ induced by the action of T^m on $\mathcal{Z}_K$, and its orbit space admits the face structure of the cone over the barycentric subdivision of K as described in [26].

Such spaces are commonly referred to as *toric spaces* or *(partial) quotients*, and are fundamental in the study of *toric topology*. Consequently, understanding which subtori H of T^m can act freely on $\mathcal{Z}_K$ is of significant importance. The Buchstaber number $s(K)$ is the maximal integer r for which there exists a subtorus of rank r acting freely on $\mathcal{Z}_K$. Similarly, taking a subgroup H of $\mathbb{Z}_2^m$ acting freely on $\mathbb{R}\mathcal{Z}_K$ yields the quotient space $\mathbb{R}\mathcal{Z}_K/H$ which is referred to as a *real toric space* or a *real (partial) quotient*. The real Buchstaber number $s_{\mathbb{R}}(K)$ is similarly defined by the existence of a subgroup acting freely on $\mathbb{R}\mathcal{Z}_K$. The determination of (real) Buchstaber numbers is challenging. We refer to the following publications for details: [25], [72], [68], [9], and [125].

It is known that the real moment-angle complex $\mathbb{R}\mathcal{Z}_K$ is the set of fixed points by the involution on $\mathcal{Z}_K$ induced by the complex conjugation on $D^2 \subset \mathbb{C}$. This implies that a T^m-action on $\mathcal{Z}_K$ induces a $\mathbb{Z}_2^m$-action on $\mathbb{R}\mathcal{Z}_K$, and then an r-dimensional subtorus of T^m acting freely on $\mathcal{Z}_K$ induces a rank r subgroup of $\mathbb{Z}_2^m$ acting freely on $\mathbb{R}\mathcal{Z}_K$. Thus, we obtain the inequality $s(K) \leqslant s_{\mathbb{R}}(K)$, and Ayzenberg [8] noted that the equality does not generally hold; specifically, there exists a simplicial complex whose real Buchstaber number is strictly bigger than its Buchstaber number.

From now on, we zero in on the case when K is a PL sphere, since in this case, all toric spaces over K are PL manifolds [27]. If K is $(n-1)$-dimensional, we have the inequalities $s(K) \leqslant s_{\mathbb{R}}(K) \leqslant m-n$. Given the condition $s(K) = m-n$, which is a special case often encountered in various fields of mathematics, the manifold $\mathcal{Z}_K/H$, for a maximal subtorus $H \subset T^m$ acting freely on $\mathcal{Z}_K$, is termed a *topological toric manifold* [95] when K is star-shaped. If K is polytopal, the manifold is referred to as a *quasitoric manifold* [60]. Note that "quasitoric manifolds" were originally called "toric manifolds" in [60], and were renamed in [25] to avoid confusion with smooth compact toric varieties. Similarly, given the condition $s_{\mathbb{R}}(K) = m-n$, the manifold $\mathbb{R}\mathcal{Z}_K/H$, for a maximal subgroup H acting freely on $\mathbb{R}\mathcal{Z}_K$, is called a *real topological toric manifold* when K is star-shaped, and it is called a *small cover* when K is polytopal. These are the real analogs of topological toric and quasitoric manifolds, respectively.

In the class of PL spheres, no examples have been known where $s(K) < s_{\mathbb{R}}(K)$. In light of this observation, one may ask whether $s(K) = s_{\mathbb{R}}(K)$ for every PL sphere K, and the following stronger question can be considered.

Problem 6.1.1. Let K be a PL sphere on $[m]$. Given a subgroup of $\mathbb{Z}_2^m$ acting freely on $\mathbb{R}\mathcal{Z}_K$, is this action induced by a subtorus of T^m acting freely on $\mathcal{Z}_K$?

In particular, when $s_{\mathbb{R}}(K) = m-n$, Problem 6.1.1 is equivalent to the (toric) *lifting problem* (Problem 6.2.8). In other words, this asks whether every small cover (or real topological toric manifold) is induced from some quasitoric manifold (or topological toric manifold, respectively). The lifting problem was initially proposed by Zhi Lü at the toric topology conference held in Osaka in 2011, as documented in [44], and remains an open problem in toric topology, attracting major research attention. In [44], Choi and Park answered positively to the lifting problem in the case $m \leqslant n+3$. However, significant advances in resolving this problem have been elusive. This chapter aims to make a contribution by providing meaningful results to the lifting problem, and more broadly to Problem 6.1.1, in the case $m \leqslant n+4$, as follows.

Theorem 6.1.2. *Let K be an $(n-1)$-dimensional PL sphere with $m \leqslant n+4$ vertices. Then, any subgroup of $\mathbb{Z}_2^m$ acting freely on $\mathbb{R}\mathcal{Z}_K$ is induced by a subtorus of T^m acting freely on $\mathcal{Z}_K$.*

The structure of this chapter is as follows. We start by some preliminaries in Section 6.2 In Section 6.3, we explain the toric wedge induction and its modification, which give a powerful framework for proving Theorem 6.1.2. This method was firstly introduced by Choi and Park in [44], and can be effectively used to demonstrate properties of toric spaces with a fixed Picard number. In Section 6.4, we prove Theorem 6.1.2. More precisely, let H be a subgroup of $\mathbb{Z}_2^m$ of rank r acting freely on $\mathbb{R}\mathcal{Z}_K$. Then, $0 \leqslant r \leqslant \min(s_{\mathbb{R}}(K), m-n) \leqslant 4$. We split the proof into two distinct cases: the case where $r \leqslant 3$ and the case where $r = s_{\mathbb{R}}(K) = m-n = 4$. In Section 6.4.1, we give a positive answer to the problem under the condition that $r \leqslant 3$ without the necessity of $m = n+4$. It demonstrates that the theorem holds for the case where $m \leqslant n+3$, or the case $m = n+4$ and $r \leqslant 3$. In Section 6.4.2 we use the modified toric wedge induction to deal with the case $r = 4$ and $m = n+4$, and prove its inductive step.

The basis step is verified in Section 6.5, in which we crucially use the combinatorial structure of the universal complex, appearing in [16, 60]. We find that, in the basis step of the toric wedge induction, every PL sphere that has less than 168 facets directly admits a lift, ruling out many cases. Finally, using symmetries, there remain 22 cases that we treat algorithmically, concluding the proof of Theorem 6.1.2.

6.2 Preliminaries

Throughout this chapter, we assume that every simplicial complex is pure. In addition, we use the following notation for describing submatrices of a given matrix. Let A be an $n \times m$ matrix over $\mathbb{Z}$. For any subsets $I \subseteq [n]$ and $J \subseteq [m]$, denote by A_I the submatrix of A formed by selecting rows indexed by $i \in I$, and by A^J the submatrix formed by selecting columns indexed by $j \in J$. Moreover, we set $I^c = [n] \setminus I$.

6.2.1 Subtorus acting on moment-angle complexes

Let K be an $(n-1)$-dimensional simplicial complex on $[m] = \{1, 2, \ldots, m\}$, $H \subset T^m$ a subtorus of dimension $r \leqslant m - n$. After choosing a basis, it can be written as

$$H = \{(e^{2\pi i(s_{11}\phi_1 + \cdots + s_{1r}\phi_r)}, \ldots, e^{2\pi i(s_{m1}\phi_1 + \cdots + s_{mr}\phi_r)}) \in T^m \mid \phi_j \in \mathbb{R}, j = 1, ..., r\}, \tag{6.2.1}$$

where $s_{ij} \in \mathbb{Z}$. We define an $m \times r$ integer matrix $S = (s_{ij})$.

The following proposition was proved for polytopal simplicial complexes [25], but it can be also proved by a similar argument for general simplicial complexes [9].

Proposition 6.2.1 ([9])**.** *Let K be a simplicial complex. Then, the subtorus* (6.2.1) *acts freely on $\mathcal{Z}_K$ if and only if for every facet I of K, the matrix S_{I^c} gives a monomorphism $\mathbb{Z}^r \longrightarrow \mathbb{Z}^{m-n}$ to a direct summand.*

A similar argument holds for $\mathbb{Z}_2^m$-actions on $\mathbb{R}\mathcal{Z}_K$ [42]. Then, the mod 2 reduction of the matrix S representing a freely acting subtorus of T^m on $\mathcal{Z}_K$ represents a freely acting subgroup of $\mathbb{Z}_2^m$ on $\mathbb{R}\mathcal{Z}_K$.

6.2.2 Characteristic and dual characteristic maps

We describe here the special case when $r = m-n$ and introduce the concept of characteristic maps and dual characteristic maps.

We use the following result:

Proposition 6.2.2 ([121, Corollary 4.1c])**.** *Let A be an integral $n \times m$ matrix of full row rank. Then, the following are equivalent:*

1. *The greatest common divisor of all $n \times n$ minors of A is* 1.
2. *For any integral vector y, there exists an integral vector z such that $Az = y$.*
3. *For every real vector x, if xA is integral, then x is integral.*

We introduce an important proposition that arises in the context of $\mathbb{Z}$*-linear Gale duality* [117]; see also [74], [26] for the classical version of Gale duality.

Proposition 6.2.3. *Let A be an integral $n \times m$ matrix whose columns generate $\mathbb{Z}^n$ as a $\mathbb{Z}$-module, and let $\overline{A}$ be an $m \times (m-n)$ matrix whose columns form a $\mathbb{Z}$-basis for the kernel of A. Then, for any n-subset I of $[m]$,* $\det(A^I) = \pm 1$ *if and only if* $\det(\overline{A}_{I^c}) = \pm 1$.

Proof. Let I be an n-subset of $[m]$ such that $\det(A^I) = \pm 1$. Then A^I is unimodular, and up to reordering the columns of A, we have

$$\begin{array}{cc} & \begin{array}{cc} I & I^c \end{array} \\ (A^I)^{-1}A = & \begin{bmatrix} \mathbf{I}_n & B \end{bmatrix}, \end{array}$$

where the line above the matrix indicates the column labels. Define

$$S = \begin{array}{c} I \\ I^c \end{array} \begin{bmatrix} B \\ -\mathbf{I}_{m-n} \end{bmatrix},$$

where the column to the left of the matrix indicates the row labels. We have $AS = 0$ and S is of full column rank, so the columns of S form a $\mathbb{Q}$-basis of the kernel of A. Moreover, since $\det(S_{I^c}) = \pm 1$, the greatest common divisor of the $(m-n) \times (m-n)$ minors of S is 1. By Proposition 6.2.2, this implies that the columns of S form a $\mathbb{Z}$-basis of the kernel of A. Therefore, there exists a unimodular matrix Q of size $(m-n) \times (m-n)$ such that $SQ = \overline{A}$. It follows that

$$\det(\overline{A}_{I^c}) = \det(S_{I^c})\det(Q) = \pm\det(-\mathbf{I}_{m-n}) = \pm 1.$$

By Proposition 6.2.2, transposing A and $\overline{A}$ swaps their roles. This proves the other implication. $\square$

We remark that A and $\overline{A}$ uniquely determine each other, up to multiplication by a unimodular matrix on the left for A and on the right for $\overline{A}$; the matrix $\overline{A}$ is sometimes called a *Gale dual* to A.

Let K be an $(n-1)$-dimensional PL sphere on $[m]$, and H an r-dimensional subtorus of T^m acting freely on $\mathcal{Z}_K$ as in (6.2.1). If $r = m-n$, then H is completely described as follows. For an $n\times m$ integer matrix λ whose columns span $\mathbb{Z}^n$, its Gale dual $\overline{\lambda}$ defined in Proposition 6.2.3 determines a subtorus H of T^m, similar to the one described in (6.2.1). By Propositions 6.2.1, 6.2.2, and 6.2.3, the following statements are equivalent:

1. For every facet $\{i_1, \dots, i_n\}$ in K, $\{\lambda^{\{i_1\}}, \dots, \lambda^{\{i_n\}}\}$ is a basis for $\mathbb{Z}^n$.
2. For every facet $\{i_1, \dots, i_n\}$ in K, $\{\overline{\lambda}_{\{i\}} \mid i \notin \{i_1, \dots, i_n\}\}$ is a basis for $\mathbb{Z}^{m-n}$.
3. The subtorus H defined by $\overline{\lambda}$ acts freely on $\mathcal{Z}_K$.

We regard λ as a map $\lambda\colon [m] \to \mathbb{Z}^n$ defined by $\lambda(i) = \lambda^{\{i\}}$. Then it is called a *characteristic map* over K if it satisfies condition (1). This condition is referred to as the *non-singularity condition* for K. This matrix can be interpreted as a linear map $\mathbb{Z}^m \to \mathbb{Z}^n$, and simultaneously, as an element in $\mathrm{Hom}(T^m, T^n)$. We also consider the map $\overline{\lambda}\colon [m] \to \mathbb{Z}^{m-n}$ defined by $\overline{\lambda}(i) = \overline{\lambda}_{\{i\}}$. This is called a *dual characteristic map* over K if it satisfies condition (2). For any subset $I \subseteq [m]$, we denote by $\lambda(I)$ and $\overline{\lambda}(I)$ the submatrices λ^I and $\overline{\lambda}_I$, respectively.

When considering the toric space $\mathcal{Z}_K/H$, the kernel of λ itself is essential whereas the choice of a basis of the kernel is not important. In this context, we introduce the concepts of *Davis-Januszkiewicz equivalence*, or simply *D-J equivalence*, for characteristic maps and dual characteristic maps. Two characteristic maps are said to be D-J equivalent if one is obtained by row operations from the other. Two dual characteristic maps are said to be D-J equivalent if one is obtained by column operations from the other. This also removes the ambiguity arising from choosing a matrix $\overline{\lambda}$.

We similarly define a *mod* 2 *characteristic map* $\lambda^{\mathbb{R}}\colon [m] \to \mathbb{Z}_2^n$ over K and a *mod* 2 *dual characteristic map* $\overline{\lambda^{\mathbb{R}}}\colon [m] \to \mathbb{Z}_2^{m-n}$ over K, to be maps satisfying conditions (1) and (2) with $\mathbb{Z}$ replaced by $\mathbb{Z}_2$. By analogy with the integral case, we also define D-J equivalence for these maps. These maps characterize subgroups of $\mathbb{Z}_2^m$ acting freely on $\mathbb{R}\mathcal{Z}_K$ in a manner analogous to the above equivalence.

6.2.3 Toric lifting problem

Observe that the mod 2 reduction of a characteristic map λ over K is a mod 2 characteristic map over K. Conversely, given a mod 2 characteristic map $\lambda^{\mathbb{R}}\colon [m] \to \mathbb{Z}_2^n$ over K and a characteristic map $\lambda\colon [m] \to \mathbb{Z}^n$ over K, if $\lambda^{\mathbb{R}}$ coincides with the composition of λ and the modulo 2 reduction map $\mathbb{Z}^n \to \mathbb{Z}_2^n$, then λ is called a *lift* of $\lambda^{\mathbb{R}}$:

$$\begin{tikzcd} & \mathbb{Z}^n \arrow[dr, "\text{mod } 2"] & \\ {[m]} \arrow[ur, dashed, "\exists\lambda"] \arrow[rr, "\lambda^{\mathbb{R}}"'] & & \mathbb{Z}_2^n. \end{tikzcd}$$

Proposition 6.2.4 ([44]). *The existence of a lift is a property of the D-J class.*

Proof. Let $\lambda^{\mathbb{R}}$ and $\mu^{\mathbb{R}}$ be two D-J equivalent mod 2 characteristic maps over K. There is an invertible matrix A over $\mathbb{Z}_2$ such that $A\lambda^{\mathbb{R}} = \mu^{\mathbb{R}}$. Suppose that $\widetilde{\lambda^{\mathbb{R}}}$ is a lift of $\lambda^{\mathbb{R}}$. There is an integer matrix $\widetilde{A}$ such that $\widetilde{A} \equiv A$ mod 2 and the determinant of $\widetilde{A}$ is odd. Then, there is an invertible integer matrix B such that $B\widetilde{\lambda^{\mathbb{R}}} \equiv \widetilde{A}\widetilde{\lambda^{\mathbb{R}}} \equiv \mu^{\mathbb{R}}$ mod 2. This implies that $B\widetilde{\lambda^{\mathbb{R}}}$ is a lift of $\mu^{\mathbb{R}}$ as well. $\square$

For convenience, we define the *dual complex* $\overline{K}$ of K as the simplicial complex whose facets are the cofacets of K. Then, by Proposition 6.2.3, $\overline{\lambda}$ is a characteristic map over $\overline{K}$. This also holds for mod 2 dual characteristic maps.

Lemma 6.2.5. *Let K be a simplicial complex. A mod 2 characteristic map $\lambda^{\mathbb{R}}$ over K has a lift if and only if $\overline{\lambda^{\mathbb{R}}}$ has a lift as a mod 2 characteristic map over $\overline{K}$.*

Proof. Suppose that $\lambda^{\mathbb{R}}$ has a lift $\widetilde{\lambda^{\mathbb{R}}}$. Then, $\overline{\widetilde{\lambda^{\mathbb{R}}}}$ is a characteristic map over $\overline{K}$. From the mod 2 reduction of the equation $\widetilde{\lambda^{\mathbb{R}}} \times \overline{\widetilde{\lambda^{\mathbb{R}}}} = \mathbf{0}$, the mod 2 reduction of the columns of $\overline{\widetilde{\lambda^{\mathbb{R}}}}$ is a basis of $\ker \lambda^{\mathbb{R}}$. Hence up to D-J equivalence, $\overline{\lambda^{\mathbb{R}}} \equiv \overline{\widetilde{\lambda^{\mathbb{R}}}}$ mod 2, that is, $\overline{\widetilde{\lambda^{\mathbb{R}}}}$ is a lift of $\overline{\lambda^{\mathbb{R}}}$.

The other direction is the same since $\overline{\overline{K}} = K$ and we can consider $\overline{\overline{\lambda^{\mathbb{R}}}} = \lambda^{\mathbb{R}}$. $\square$

Example 6.2.6. Let K be the boundary complex of a triangular bipyramid with vertex set $[5] = \{1,2,3,4,5\}$ and facet set $\{123, 124, 134, 235, 245, 345\}$, where the concatenated numbers represent sets of vertices. Then, its dual complex $\overline{K}$ has facet set $\{12, 13, 14, 25, 35, 45\}$.

We consider a mod 2 characteristic map $\lambda^{\mathbb{R}}$ and a mod 2 dual characteristic map $\overline{\lambda^{\mathbb{R}}}$ over K as follows:

$$\lambda^{\mathbb{R}} = \begin{array}{c} \begin{array}{ccccc} \scriptstyle 1 & \scriptstyle 2 & \scriptstyle 3 & \scriptstyle 4 & \scriptstyle 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}, \quad \textit{and} \quad \overline{\lambda^{\mathbb{R}}} = \begin{array}{c} \scriptstyle 1 \\ \scriptstyle 2 \\ \scriptstyle 3 \\ \scriptstyle 4 \\ \scriptstyle 5 \end{array} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

If we regard the matrices as integer matrices, then they can be seen as their own lifts.

If a lift λ of $\lambda^{\mathbb{R}}$ sends $[m]$ to $\{0,1\}$-vectors, then it is called a $\{0,1\}$-*lift* of $\lambda^{\mathbb{R}}$. Similarly, it is called a $\{0,\pm 1\}$-*lift* when it sends $[m]$ to $\{0,\pm 1\}$-vectors.

Example 6.2.7. Let K be the join $\partial\Delta^{n_1} * \partial\Delta^{n_2} * \cdots * \partial\Delta^{n_p}$ of the boundaries of p simplices. We denote its set of vertices as

$$\{i_j \mid 1 \leqslant i \leqslant p, 1 \leqslant j \leqslant n_i + 1\},$$

where i_1, i_2, $\ldots$, i_{n_i+1} comes from the vertices of $\partial\Delta^{n_i}$. Let $F^i_j = \{i_1, i_2, \ldots, i_{n_j+1}\} \setminus \{i_j\}$. The set of facets of K is

$$\{\cup_{i=1}^p F^i_{j_i} \mid 1 \leqslant j_i \leqslant n_i + 1\}.$$

By [43], up to D-J equivalence and vertex relabeling, a mod 2 characteristic map over K is of the form

$$\lambda^{\mathbb{R}} = \begin{bmatrix} \lambda^{\mathbb{R}}_1 & & & & \\ * & \lambda^{\mathbb{R}}_2 & & & \\ \vdots & * & \ddots & & \\ & \vdots & & \lambda^{\mathbb{R}}_{p-1} & \\ * & * & & * & \lambda^{\mathbb{R}}_p \end{bmatrix},$$

where $\lambda^{\mathbb{R}}_i$ is a mod 2 characteristic map over $\partial\Delta^{n_i}$ and the empty spaces display zeros. Up to D-J equivalence and vertex relabeling,

$$\lambda^{\mathbb{R}}_i = \begin{array}{c} \begin{array}{cc} i_1 \cdots i_{n_i} & i_{n_i+1} \end{array} \\ \left[\begin{array}{cc} & 1 \\ \mathbf{I}_{n_i} & \vdots \\ & 1 \end{array}\right] \end{array}.$$

Let $\widetilde{\lambda^{\mathbb{R}}}$ and $\widetilde{\lambda^{\mathbb{R}}_i}$ be the $\{0,1\}$-matrices over $\mathbb{Z}$ such that $\widetilde{\lambda^{\mathbb{R}}} \equiv \lambda^{\mathbb{R}} \bmod 2$ and $\widetilde{\lambda^{\mathbb{R}}_i} \equiv \lambda^{\mathbb{R}}_i \bmod 2$, for $i = 1, \ldots, p$, respectively. Note that $\widetilde{\lambda^{\mathbb{R}}_i}(F^i_j)$ is the submatrix obtained by removing the column indexed by i_j, and $\det(\widetilde{\lambda^{\mathbb{R}}_i}(F^i_j)) = \pm 1$. Then, for a facet $\sigma = \cup_{i=1}^p F^i_{j_i}$ of K,

$$\det(\widetilde{\lambda^{\mathbb{R}}}(\sigma)) = \det(\widetilde{\lambda^{\mathbb{R}}_i}(F^1_{j_1})) \times \cdots \times \det(\widetilde{\lambda^{\mathbb{R}}_i}(F^p_{j_p})) = \pm 1.$$

Hence $\widetilde{\lambda^{\mathbb{R}}}$ is the $\{0,1\}$-lift of $\lambda^{\mathbb{R}}$, and it shows that every mod 2 characteristic map over the join of the boundaries of simplices has the $\{0,1\}$-lift.

Since every mod 2 characteristic map over the specific PL sphere in Example 6.2.7 admits a lift, it is natural to ask whether this property holds more generally. This leads to the following question, known as the *(toric) lifting problem*.

Problem 6.2.8 ((toric) Lifting problem)**.** Let K be a PL sphere. Does every mod 2 characteristic map over K have a lift? Equivalently, does every mod 2 dual characteristic map over K have a lift when seen as a mod 2 characteristic map over $\overline{K}$?

6.2.4 Wedge operations

Let K be a simplicial complex on $[m]$. For another simplicial complex L on a disjoint vertex set from K, the *join* $K * L$ of K and L is defined as the simplicial complex

$$K * L := \{\sigma \cup \tau \mid \sigma \in K, \tau \in L\}.$$

The *suspension* of K is given by

$$\Sigma(K) := \partial I * K,$$

where I is a 1-simplex with two new vertices v_1 and v_2, and ∂I is its boundary complex. In $\Sigma(K)$, the pair $\{v_1, v_2\}$ is referred to as a *suspended pair*, and each vertex in it is called a *suspended vertex*. Note that in Example 6.2.6, K is the suspension of the boundary of a triangle with suspended pair $\{1, 5\}$.

The *link* of a face σ of K is the simplicial complex defined by

$$\mathrm{lk}_K(\sigma) := \{\tau \in K \mid \sigma \cup \tau \in K, \sigma \cap \tau = \varnothing\}.$$

The *wedge* of K at a vertex v of K is defined as

$$\mathrm{wed}_v(K) := (I * \mathrm{lk}_K(v)) \cup (\partial I * (K \setminus v)),$$

where I is the 1-simplex with two vertices, and $K \setminus v$ is the induced subcomplex of K on the vertex set $[m] \setminus \{v\}$. It is evident that the link of a new vertex added after applying a wedge to K, that is, a vertex in I, is isomorphic to K. In that sense, we often use v_1 and v_2 to refer to the two copies of v in $\mathrm{wed}_v(K)$. Consequently, $\mathrm{wed}_v(K)$ has vertex set $([m] \setminus \{v\}) \cup \{v_1, v_2\}$. Here, two vertices v_1 and v_2 are referred to as *wedged vertices* of v, and the edge connecting them as the *wedged edge* of v. Notably, if we add a ghost vertex to K, then $\Sigma(K)$ can be viewed as the wedge of K at this ghost vertex.

The wedge operation can be defined equivalently as an easy combinatorial operation on the minimal non-faces of K: we duplicate the vertex v in each minimal non-face of K it appears in. More precisely, let $\eta \subset [m]$ be a subset of the vertex set of K.

1. If η contains v, then η is a minimal non-face of K if and only if $\eta \setminus \{v\} \cup \{v_1, v_2\}$ is a minimal non-face of $\mathrm{wed}_v(K)$.
2. If η does not contain v, then η is a minimal non-face of K if and only if η is a minimal non-face of $\mathrm{wed}_v(K)$.

As for suspensions, one can easily prove that the minimal non-faces of $K_1 * K_2$ is the union of the sets of minimal non-faces of K_1 and K_2. Then, the set of minimal non-faces of $\partial I * K$ is obtained by adding I in the set of minimal non-faces of K. If we add a ghost vertex to K, which is a minimal non-face of K by itself, then the minimal non-faces of $\partial I * K$ is the minimal non-faces of the wedge at the ghost vertex. With this perspective, two consecutive wedge operations and join operations, including suspension operations, are associative and commutative with appropriate vertex identification.

Conversely, suppose that there are two vertices v_1 and v_2 such that for every minimal non-face η of K, either $\{v_1, v_2\} \subseteq \eta$ or $\{v_1, v_2\} \cap \eta = \varnothing$. If σ is a facet of K containing neither $\{v_1\}$ nor $\{v_2\}$, then $\{v_1\} \cup \sigma$ is a non-face, so there is a minimal non-face η of K containing v_1. This contradicts with the assumption. Hence every facet of K contains v_1 or v_2. By the following lemma, if $\{v_1, v_2\}$ is not a minimal non-face of K, then it is a wedged edge of K, and otherwise, it is a suspended pair of K.

Lemma 6.2.9. *[46] Let K be a PL sphere, and v_1, v_2 be two vertices of K. If every facet of K contains v_1 or v_2, then K equals to either $\Sigma(L)$ with suspended pair $\{v_1, v_2\}$, or to $\mathrm{wed}_v(L)$ with the wedged edge $\{v_1, v_2\}$ for some lower dimensional PL sphere L.*

Corollary 6.2.10. *Let $\overline{\lambda^{\mathbb{R}}}$ be a mod 2 characteristic map over a PL sphere K. If $\overline{\lambda^{\mathbb{R}}}(v_1) = \overline{\lambda^{\mathbb{R}}}(v_2)$ for two vertices v_1 and v_2 of K, then $\{v_1, v_2\}$ is a suspended pair or a wedged edge of K.*

Proof. By the non-singularity of $\overline{\lambda^{\mathbb{R}}}$, every facet of K contains v_1 or v_2 and the results comes from Lemma 6.2.9. ☐

In light of the associative and commutative nature of wedge operations, we introduce the notation $K(J)$, termed a *J-construction* of K in [14], for a positive integer m-tuple $J = (j_1, j_2, \ldots, j_m)$. This represents the simplicial complex obtained by applying multiple wedge operations to K; for each $i \in [m]$, wedge operations are applied $j_i - 1$ times to K at i or its copied vertices. We will often denote the copied vertices of i by $i_1, i_2, \ldots, i_{j_i}$. For the sake of convenience, when $j_i = 1$, we treat i either as i_1 or as i, depending on the context.

Example 6.2.11. For convenience, we represent a set of vertices by concatenated numbers. We split the facets of a wedge at v into three sets: the ones containing the new edge v_1v_2 (1), the ones containing only v_1 (2), and the ones containing only v_2 (3). Moreover, we write in bold case the two new vertices appearing after each wedge is performed. Let P_5 be a pentagon on [5]. We compute the facets of $\partial P_5(2,2,1,1,1)$ by computing two wedges. The facets of ∂P_5 is represented on the left of Figure 6.1 and are $\{\underline{12}, \underline{15}, 23, 34, 45\}$, where the ones underlined contain the vertex 1. Its set of minimal non-faces is $\{13, 14, 24, 25, 35\}$. The set of facets of $\partial P_5(2,1,1,1,1) = \mathrm{wed}_1(\partial P_5)$ is shown on the right of Figure 6.1. It is divided into the following three parts, where the facets containing the vertex 2 are underlined.

1. $\{\underline{\mathbf{1_1 1_2}2}, \mathbf{1_1 1_2}5\}$,
2. $\{\underline{\mathbf{1_1}23}, \mathbf{1_1}34, \mathbf{1_1}45\}$,
3. $\{\underline{\mathbf{1_2}23}, \mathbf{1_2}34, \mathbf{1_2}45\}$.

Its set of minimal non-faces is $\{\mathbf{1_1 1_2}3, \mathbf{1_1 1_2}4, 24, 25, 35\}$.

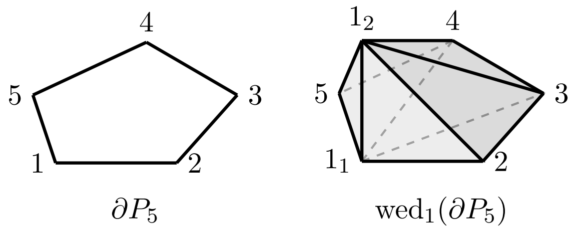


Figure 6.1 – Wedge at the vertex 1 of the boundary of a pentagon.

The set of facets of $\partial P_5(2,2,1,1,1) = \mathrm{wed}_2(\mathrm{wed}_1(\partial P_5))$ is split into

1. $\{1_1 1_2 \mathbf{2_1 2_2}, 1_1 \mathbf{2_1 2_2} 3, 1_2 \mathbf{2_1 2_2} 3\}$,
2. $\{\mathbf{2_1} 1_1 1_2 5, \mathbf{2_1} 1_1 34, \mathbf{2_1} 1_1 45, \mathbf{2_1} 1_2 34, \mathbf{2_1} 1_2 45\}$, and
3. $\{\mathbf{2_2} 1_1 1_2 5, \mathbf{2_2} 1_1 34, \mathbf{2_2} 1_1 45, \mathbf{2_2} 1_2 34, \mathbf{2_2} 1_2 45\}$.

Its set of minimal non-faces is $\{1_1 1_2 3, 1_1 1_2 4, \mathbf{2_1 2_2} 4, \mathbf{2_1 2_2} 5, 35\}$.

Due to the commutativity and associativity of the operations involved, we have the relationship:

$$(\partial I * K)(j_1, j_2, \ldots, j_{m+2}) = \partial I(j_1, j_2) * K(j_3, \ldots, j_{m+2}). \tag{6.2.2}$$

This leads to two characterizations regarding suspensions and wedges.

Proposition 6.2.12. *Let K be a simplicial complex. Then*

1. *K is a suspension if and only if so is $\mathrm{wed}_v(K)$ for any non-suspended vertex v of K,*
2. *K is a wedge if and only if so is $\Sigma(K)$.*

We define the *Picard number* of K as $m - n$. One can observe that the wedge operation preserves the Picard number of K while the suspension increases the Picard number of K by 1. It is known that the link, wedge, and suspension operations are closed within the class of PL spheres, see [39] for details.

A PL sphere not isomorphic to the wedge of another PL sphere is called a *seed*. It should be noted that any PL sphere K with Picard number p can be written as $L(J)$, where L is a seed of the same Picard number p. In addition, one can easily see that L is uniquely determined up to isomorphism, whereas J can be different.

For our purpose, we are interested in PL spheres of dimension $n-1$ on $[m]$ whose real Buchstaber number coincides with their Picard number $m-n$. Such PL spheres are called *($\mathbb{Z}_2^n$-)colorable*. Ewald [69] observed that all colorable PL spheres are obtained by colorable seeds, and Choi and Park [46] proved that the number of colorable seeds with given Picard number is finite. Although obtaining the list of colorable seeds of given Picard number is a difficult problem in itself, the list up to Picard number 4 has been established in [39], see Theorem 4.1.2.

Theorem 6.2.13 ([39]). *The number of colorable seeds with Picard number at most 4 up to isomorphism is as follows:*

	n											
p	1	2	3	4	5	6	7	8	9	10	11	>11	*total*
1	1												1
2		1											1
3		1	1	1									3
4		1	4	21	142	733	1190	776	243	39	4		3153

with the empty slots displaying zero.

Note that Theorem 6.2.13 comes together with a database containing the explicit description (the set of facets) of every toric colorable seed with Picard number at most 4. We will leverage this database in Section 6.5.

6.3 Toric wedge induction

6.3.1 Toric wedge induction and its modification

Let K be an $(n-1)$-dimensional PL sphere on $[m]$. The cornerstone of toric wedge induction is that the set of all mod 2 characteristic maps over K determines the mod 2 characteristic maps over a PL sphere obtained from a link or a wedge of K.

Let $\lambda^{\mathbb{R}}$ be a mod 2 characteristic map over K. The *projection* of $\lambda^{\mathbb{R}}$ with respect to a face σ of K is a map from the vertex set of $\mathrm{lk}_K(\sigma)$ to $\mathbb{Z}_2^{n-|\sigma|}$ defined by

$$\mathrm{Proj}_\sigma(\lambda^{\mathbb{R}})(w) = [\lambda^{\mathbb{R}}(w)] \in \mathbb{Z}_2^n \,/\, \langle \lambda^{\mathbb{R}}(v) \mid v \in \sigma \rangle \cong \mathbb{Z}_2^{n-|\sigma|} \tag{6.3.1}$$

for each vertex w of $\mathrm{lk}_K(\sigma)$. If we fix a basis of $\mathbb{Z}_2^{n-|\sigma|}$, we can see that $\mathrm{Proj}_\sigma(\lambda^{\mathbb{R}})$ is a mod 2 characteristic map over $\mathrm{lk}_K(\sigma)$. We also call $\mathrm{Proj}_\sigma(\lambda^{\mathbb{R}})$ the *projection* of $\lambda^{\mathbb{R}}$ onto $\mathrm{lk}_K(\sigma)$.

Let $\Lambda^{\mathbb{R}}$ be a mod 2 characteristic map over $\mathrm{wed}_v(K)$, for v some vertex of K. Up to D-J equivalence and vertex relabeling, we may assume that

$$\Lambda^{\mathbb{R}} = \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & \phantom{\mathbf{I}_{n-1}} & \end{array} \\ \begin{bmatrix} 1 & 0 & \mathbf{0} & \mathbf{a} \\ 0 & 1 & \mathbf{0} & \mathbf{b} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{n-1} & A \end{bmatrix} \end{array}, \tag{6.3.2}$$

where the column indexes v_1 and v_2 stand for the associated wedged vertices, $\mathbf{a}$ and $\mathbf{b}$ are row vectors of size $m-n$, $\mathbf{I}_{n-1}$ is the identity matrix of size $n-1$, and A is a $\mathbb{Z}_2$-matrix of size $(n-1)\times(m-n)$. The links $\mathrm{lk}_{\mathrm{wed}_v(K)}(v_1)$ and $\mathrm{lk}_{\mathrm{wed}_v(K)}(v_2)$ are isomorphic to K by identifying v_2 and v_1 with v, respectively, and the projections of $\Lambda^{\mathbb{R}}$ with respect to v_1 and v_2 are written as

$$\lambda_1^{\mathbb{R}} = \mathrm{Proj}_{v_1}(\Lambda^{\mathbb{R}}) = \begin{array}{c} v_2 \\ \left[\begin{array}{ccc} 1 & \mathbf{0} & \mathbf{b} \\ \mathbf{0} & \mathbf{I}_{n-1} & A \end{array}\right] \end{array}, \text{ and}$$

$$\lambda_2^{\mathbb{R}} = \mathrm{Proj}_{v_2}(\Lambda^{\mathbb{R}}) = \begin{array}{c} v_1 \\ \left[\begin{array}{ccc} 1 & \mathbf{0} & \mathbf{a} \\ \mathbf{0} & \mathbf{I}_{n-1} & A \end{array}\right] \end{array}.$$

If we consider that the first column of each matrix corresponds to the vertex v, then both matrices $\lambda_1^{\mathbb{R}}$ and $\lambda_2^{\mathbb{R}}$ are mod 2 characteristic maps over K. Hence, $\Lambda^{\mathbb{R}}$ corresponds to a choice of two mod 2 characteristic maps over K whose first n columns form an identity matrix such that their submatrices formed by deleting the first row and the first column are identical up to D-J equivalence.

Conversely, one may construct at most one $\Lambda^{\mathbb{R}}$ over $\mathrm{wed}_v(K)$ from an ordered pair of mod 2 characteristic maps $\lambda_1^{\mathbb{R}}$ and $\lambda_2^{\mathbb{R}}$ over K as in (6.3.2). We denote $\Lambda^{\mathbb{R}} = \lambda_1^{\mathbb{R}} \wedge_v \lambda_2^{\mathbb{R}}$ if it exists. If $\lambda_1^{\mathbb{R}} = \lambda_2^{\mathbb{R}} = \lambda^{\mathbb{R}}$, then $\lambda^{\mathbb{R}} \wedge_v \lambda^{\mathbb{R}}$ always exists for any v, and it is called the *canonical extension* of $\lambda^{\mathbb{R}}$ at v.

It should be noted that we can represent each characteristic map over $K(J)$ by a combination of characteristic maps over K. From this viewpoint, we shall introduce one powerful inductive tool to demonstrate some properties on real toric spaces, for example the existence of a lift of $\Lambda^{\mathbb{R}}$ as in the present chapter.

For a PL sphere K, let $\mathcal{X}$ be a collection of pairs $(L, \lambda^{\mathbb{R}})$ such that every L is expressed as $K(J)$ for some J, and $\lambda^{\mathbb{R}}$ is a mod 2 characteristic map over L. Then, $\mathcal{X}$ is called a *wedge-stable set based on* K if $(L, \Lambda^{\mathbb{R}}) \in \mathcal{X}$ implies that there exist $(L', \lambda_1^{\mathbb{R}})$ and $(L', \lambda_2^{\mathbb{R}})$ in $\mathcal{X}$ such that $L = \mathrm{wed}_v(L')$ for a vertex v of L' and $\Lambda^{\mathbb{R}} = \lambda_1^{\mathbb{R}} \wedge_v \lambda_2^{\mathbb{R}}$.

We present the concept of *toric wedge induction* which is a method employed to demonstrate the validity of a given property across $\mathcal{X}$.

Proposition 6.3.1 (Toric wedge induction)**.** *For a PL sphere K, let $\mathcal{X}$ be a wedge-stable set based on K, and $\mathcal{P}$ a property. Suppose that the following holds;*

1. ***Basis step:*** *Every $(K, \lambda^{\mathbb{R}}) \in \mathcal{X}$ satisfies $\mathcal{P}$.*
2. ***Inductive step:*** *If $(L, \lambda_1^{\mathbb{R}})$, $(L, \lambda_2^{\mathbb{R}}) \in \mathcal{X}$ satisfy $\mathcal{P}$, then so does $(\mathrm{wed}_v(L), \lambda_1^{\mathbb{R}} \wedge_v \lambda_2^{\mathbb{R}})$ for every vertex v of L such that $\lambda_1^{\mathbb{R}} \wedge_v \lambda_2^{\mathbb{R}}$ exists.*

Then, $\mathcal{P}$ holds on $\mathcal{X}$.

The credit for the original idea behind toric wedge induction should be given to Choi and Park [44]. They used it for showing the projectivity of certain toric manifolds in [44] or [45]. Later, the authors of this paper used it for classifying toric manifolds satisfying equality within an inequality regarding the number of minimal components in their rational curve space [39]. However, the inductive step can sometimes be challenging. To address this, we can relax it by strengthening the basis step. In this chapter, we introduce a refined version of this approach that simplifies the argument and facilitates the proof of the toric lifting property we aim to establish.

First, we briefly review some required notions and properties. We refer to [46], where the reader may find much more details about the relations between the characteristic maps over $K(J)$ and the puzzles explained below.

The *pre-diagram* $D'(K)$ of K is an edge-colored non-simple graph such that

1. the node set of $D'(K)$ is the set of D-J classes of the mod 2 characteristic maps over K, and
2. for a vertex v of K and two mod 2 characteristic maps $\lambda_1^{\mathbb{R}}$ and $\lambda_2^{\mathbb{R}}$ over K, a pair $(\{\lambda_1^{\mathbb{R}}, \lambda_2^{\mathbb{R}}\}, v)$ is a colored edge of $D'(K)$ if and only if there is a mod 2 characteristic map over $\mathrm{wed}_v(K)$ whose two projections onto K are $\lambda_1^{\mathbb{R}}$ and $\lambda_2^{\mathbb{R}}$.

We denote $G(J)$ the 1-skeleton of the simple polytope $P(J) := \Delta^{j_1-1} \times \Delta^{j_2-1} \times \cdots \times \Delta^{j_m-1}$. Each edge $\boldsymbol{\epsilon}$ of $G(J)$ is uniquely written as

$$\boldsymbol{\epsilon} = \alpha_1 \times \alpha_2 \times \cdots \times \alpha_{v-1} \times \epsilon_v \times \alpha_{v+1} \times \cdots \times \alpha_m,$$

where α_i is a vertex of Δ^{j_i-1}, $1 \leqslant i \leqslant m, i \neq v$, and ϵ_v is an edge of Δ^{j_v-1}. Then, color $\boldsymbol{\epsilon}$ by v, and we call such edge a v-edge of $G(J)$.

Then, a mod 2 characteristic map $\lambda^{\mathbb{R}}$ over $K(J)$ can be expressed by an edge-colored graph homomorphism $\phi\colon G(J) \to D'(K)$. When $\boldsymbol{\alpha}$ is a vertex of $G(J)$, we can write

$$\boldsymbol{\alpha} = \alpha_1 \times \alpha_2 \times \cdots \times \alpha_m,$$

where $1 \leqslant \alpha_i \leqslant j_i$ is a vertex of Δ^{j_i-1} for $1 \leqslant i \leqslant m$. Observe that

$$\mathcal{F}(\boldsymbol{\alpha}) := \{1_1, 1_2, \ldots, 1_{j_1}, \ldots, m_1, m_2, \ldots, m_{j_m}\} \setminus \{1_{\alpha_1}, 2_{\alpha_2}, \ldots, m_{\alpha_m}\}$$

does not contain any minimal non-face of $K(J)$, so it is a face of $K(J)$. Since each vertex i_{j_k} ($j_k \neq \alpha_i$) in $\mathcal{F}(\boldsymbol{\alpha})$ is a wedged vertex of i_{α_i}, $\mathrm{lk}_{K(J)}(\mathcal{F}(\boldsymbol{\alpha}))$ is isomorphic to K, and the projection $\mathrm{Proj}_{\mathcal{F}(\boldsymbol{\alpha})}(\lambda^{\mathbb{R}})$ is a mod 2 characteristic map over K by the natural bijection between $\{1, 2, \ldots, m\}$ and $\{1_{\alpha_1}, 2_{\alpha_2}, \ldots, m_{\alpha_m}\}$. Define ϕ by $\phi(\boldsymbol{\alpha}) = \mathrm{Proj}_{\mathcal{F}(\boldsymbol{\alpha})}(\lambda^{\mathbb{R}})$ for each vertex $\boldsymbol{\alpha}$ of $G(J)$. Let v be a vertex of K and $\boldsymbol{\epsilon}$ a v-edge of $G(J)$. It consists of two vertices $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha'}$ that coincide everywhere except at $\alpha_v \neq \alpha'_v$, with $\alpha_v, \alpha'_v \in \Delta^{j_v-1}$. Then $\phi(\boldsymbol{\epsilon}) = (\{\mathrm{Proj}_{\mathcal{F}(\boldsymbol{\alpha})}(\lambda^{\mathbb{R}}), \mathrm{Proj}_{\mathcal{F}(\boldsymbol{\alpha'})}(\lambda^{\mathbb{R}})\}, v)$. By the following proposition, $\overline{\mathrm{Proj}_{\mathcal{F}(\boldsymbol{\alpha})}(\lambda^{\mathbb{R}})}$ and $\overline{\mathrm{Proj}_{\mathcal{F}(\boldsymbol{\alpha'})}(\lambda^{\mathbb{R}})}$ coincide everywhere except possibly at the rows corresponding to v_{α_v} and $v_{\alpha'_v}$, respectively. Hence ϕ is an edge-colored graph homomorphism.

Proposition 6.3.2. *Let K be a PL sphere, and σ a face of K such that the Picard numbers of K and $\mathrm{lk}_K(\sigma)$ are the same. Then, up to D-J equivalence, the dual of the projection of a mod 2 characteristic map $\lambda^{\mathbb{R}}$ with respect to a face σ of K is obtained by removing the rows corresponding to σ in $\overline{\lambda^{\mathbb{R}}}$.*

Proof. It is sufficient to prove the result for a vertex $\sigma = \{v\}$. To project $\lambda^{\mathbb{R}}$ with respect to $\{v\}$, let us left multiply $\lambda^{\mathbb{R}}$ by an invertible matrix g so that the vector $g\lambda^{\mathbb{R}}(v)$ has a single nonzero entry, say at index k. From (6.3.1), if we delete the vth column and kth row of $g\lambda^{\mathbb{R}}$, then we obtain the projection $\mathrm{Proj}_v(\lambda^{\mathbb{R}})$. Let $\overline{\mu^{\mathbb{R}}}$ be the matrix obtained by removing vth row in $\overline{\lambda^{\mathbb{R}}}$. Then, any column of $\overline{\mu^{\mathbb{R}}}$ is an element of the kernel of $\mathrm{Proj}_v(\lambda^{\mathbb{R}})$ since every component of the vth column of $\lambda^{\mathbb{R}}$ except the kth one is 0. Note that $\overline{\mu^{\mathbb{R}}}$ remains of full rank since there must exist a cofacet of K that does not contain v. □

However, not all edge-colored graph homomorphism ϕ is obtained from a mod 2 characteristic map over $K(J)$. If it is, ϕ is called a *(realizable) puzzle* over $K(J)$ (or, over K, if there is no confusion), and $\lambda_\phi^{\mathbb{R}}$ denotes the corresponding mod 2 characteristic map over $K(J)$. A puzzle is called *irreducible* if it does not contain any edge corresponding to a canonical extension; otherwise, it is called *reducible*.

Remark 6.3.3. Let ϕ be a realizable puzzle over $K(J)$. When a v-edge consists of two vertices $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha'}$, it is worthy to note that if $\overline{\lambda_\phi^{\mathbb{R}}}(v_{\alpha_v}) = \overline{\lambda_\phi^{\mathbb{R}}}(v_{\alpha'_v})$, then ϕ is not irreducible.

Consider a realizable puzzle ϕ over $K(J)$. In $G(J)$, two v-edges

$$\begin{aligned}\boldsymbol{\epsilon} &= \alpha_1 \times \alpha_2 \times \cdots \times \alpha_{v-1} \times \epsilon_v \times \alpha_{v+1} \times \cdots \times \alpha_m,\\ \boldsymbol{\epsilon}' &= \alpha_1' \times \alpha_2' \times \cdots \times \alpha_{v-1}' \times \epsilon_v' \times \alpha_{v+1}' \times \cdots \times \alpha_m'\end{aligned}$$

are called *parallel* if $\epsilon_v = \epsilon_v'$. By [46, Corollary 4.4], if there is a v-edge $\boldsymbol{\epsilon}$ of ϕ corresponding to a canonical extension, then so does every edge parallel to $\boldsymbol{\epsilon}$. Therefore, every puzzle is obtainable from an irreducible puzzle by a sequence of canonical extensions.

Proposition 6.3.4 (Modified toric wedge induction)**.** *For a PL sphere K, let $\mathcal{X}$ be a wedge-stable set based on K, and $\mathcal{P}$ a property. Suppose that the following holds;*

1. ***Basis step:*** *For every positive integer tuple J and every irreducible realizable puzzle ϕ over $K(J)$, $(K(J), \lambda^{\mathbb{R}}_\phi)$ satisfies $\mathcal{P}$.*
2. ***Inductive step:*** *If $(L, \lambda^{\mathbb{R}}) \in \mathcal{X}$ satisfies $\mathcal{P}$, then so does the pair consisting of the wedge of L at v and the canonical extension of $\lambda^{\mathbb{R}}$ at v for any vertex v of L.*

Then, $\mathcal{P}$ holds on $\mathcal{X}$.

It should be noted that the basis step consists of finitely many cases as shown in the lemma below.

Lemma 6.3.5. *For a simplicial complex K, there are finitely many irreducible puzzles over K.*

Proof. Let $J = (j_1, j_2, \ldots, j_m)$. Fix a vertex $\boldsymbol{\alpha} = \alpha_1 \times \alpha_2 \times \cdots \times \alpha_m$ of $P(J)$. For each j_v, $\boldsymbol{\alpha}$ is also a vertex of the simplex $\alpha_1 \times \alpha_2 \times \cdots \times \alpha_{v-1} \times \Delta^{j_v - 1} \times \alpha_{v+1} \times \cdots \times \alpha_m$. Because any two vertices of Δ^{j_v-1} forms an edge, the number j_v can not exceed the number of mod 2 characteristic maps over K. □

Remark 6.3.6. The concept of wedge of characteristic maps and puzzle is not only described for mod 2 characteristic maps, but also for characteristic maps. See [46] for details. This means that we can apply toric wedge induction to a collection of toric spaces. In addition, the number of PL spheres which admit a characteristic map is less than or equal to the number of PL spheres which admit a mod 2 characteristic map, so it is finite. However, the number of characteristic maps over a seed is not finite. Hence the basis step may not be implemented by direct computations in finite steps.

6.3.2 Modified Toric wedge induction in terms of dual characteristic maps

Even if the modified version (Proposition 6.3.4) of the toric wedge induction has an easier inductive step than the original version (Proposition 6.3.1), there still remains a challenging part to deal with: constructing (irreducible) puzzles. In this subsection, we characterize irreducible puzzles in terms of dual characteristic maps over seeds, and restate the modified toric wedge induction based on a seed using dual characteristic maps instead of irreducible puzzles.

Let K be a colorable seed of dimension $n-1$ on $[m]$ and $\lambda^{\mathbb{R}}$ a mod 2 characteristic map over K. Assume that there are two vertices of K such that $\overline{\lambda^{\mathbb{R}}}(v) = \overline{\lambda^{\mathbb{R}}}(w)$. By Corollary 6.2.10, K must be a suspension. Then, by Proposition 6.2.12, K is the suspension of a seed; in this case it is called a *suspended seed*. Hence, if K is a non-suspended seed, then every dual characteristic map over K is injective. An *IDCM* refers to an injective mod 2 dual characteristic map. By the following lemma, every irreducible puzzle over a non-suspended seed corresponds to an IDCM.

Lemma 6.3.7. *Let ϕ be an irreducible puzzle over $K(J)$ for a PL sphere K on $[m]$ and a positive integer m-tuple J. Then, $\overline{\lambda^{\mathbb{R}}_\phi}$ is injective if and only if $\overline{\phi(\boldsymbol{\alpha})}$ is injective for every vertex $\boldsymbol{\alpha}$ of $G(J)$.*

Proof. Assume that $\overline{\lambda^{\mathbb{R}}_{\phi}}$ is injective. By Proposition 6.3.2, each projection $\overline{\phi(\boldsymbol{\alpha})}$ of $\overline{\lambda^{\mathbb{R}}_{\phi}}$ has no repeated rows, so it is injective.

Conversely, suppose that $\overline{\lambda^{\mathbb{R}}_{\phi}}(v) = \overline{\lambda^{\mathbb{R}}_{\phi}}(w)$, for distinct vertices v and w of $K(J)$. By Remark 6.3.3 and the irreducibility of ϕ, the vertices v and w cannot be copies of the same vertex of K after wedge operations. Hence, there is a vertex $\boldsymbol{\alpha}$ of $G(J)$ such that $v = l_{\alpha_l}$ and $w = l'_{\alpha_{l'}}$ for two indexes $1 \leqslant \alpha_l \leqslant j_l$ and $1 \leqslant \alpha_{l'} \leqslant j_{l'}$ and two distinct vertices $l, l' \in [m]$. This yields $\overline{\mathrm{Proj}_{\mathcal{F}(\boldsymbol{\alpha})}(\lambda^{\mathbb{R}}_{\phi})}(v) = \overline{\mathrm{Proj}_{\mathcal{F}(\boldsymbol{\alpha})}(\lambda^{\mathbb{R}}_{\phi})}(w)$, so $\overline{\phi(\boldsymbol{\alpha})}$ is not injective. $\square$

In general, a seed K is of the form $\partial I_1 * \cdots * \partial I_q * L$, where each I_k is the 1-simplex on the vertex set $\{2k-1, 2k\}$, for $1 \leqslant k \leqslant q$, and L is a non-suspended seed. By (6.2.2), we have

$$K(J) = \partial I_1(J_1) * \cdots * \partial I_q(J_q) * L(J_{q+1}), \tag{6.3.3}$$

for some pairs of positive integers $J_1, \ldots, J_q$ and a tuple of positive integers J_{q+1}. Before studying mod 2 characteristic maps over $K(J)$, we need the following analysis of mod 2 characteristic maps over the join of two simplicial complexes. Research on the structure of characteristic maps over the join $K_1 * K_2$ is well-established in the literature such as in [48] and [63], to which we refer the reader for further information. These results are easily translated to mod 2 characteristic maps. Consider the join of simplicial complexes K_1 of dimension $n_1 - 1$ with m_1 vertices and K_2 of dimension $n_2 - 1$ with m_2 vertices. One can observe that any (mod 2) characteristic map $\lambda^{\mathbb{R}}$ over $K_1 * K_2$ has the following form;

$$\lambda^{\mathbb{R}} = \begin{bmatrix} \lambda^{\mathbb{R}}_{11} & \lambda^{\mathbb{R}}_{12} \\ \lambda^{\mathbb{R}}_{21} & \lambda^{\mathbb{R}}_{22} \end{bmatrix},$$

where $\lambda^{\mathbb{R}}_{11}$ and $\lambda^{\mathbb{R}}_{22}$ are mod 2 characteristic maps over K_1 and K_2, respectively, see [48, Lemma 3.1] for instance. Moreover, we can assume that the first n_1 columns of $\lambda^{\mathbb{R}}_{11}$, and the first n_2 columns of $\lambda^{\mathbb{R}}_{22}$ form identity matrices by D-J equivalence and vertex relabeling. Then, up to D-J equivalence,

$$\lambda^{\mathbb{R}} = \begin{bmatrix} \mathbf{I}_{n_1} & A & \mathbf{0} & B \\ \mathbf{0} & C & \mathbf{I}_{n_2} & D \end{bmatrix} \text{ and } \overline{\lambda^{\mathbb{R}}} = \begin{bmatrix} A & B \\ \mathbf{I}_{m_1-n_1} & \mathbf{0} \\ C & D \\ \mathbf{0} & \mathbf{I}_{m_2-n_2} \end{bmatrix}. \tag{6.3.4}$$

We call this form of $\lambda^{\mathbb{R}}$ the *joining representative* of the D-J class of $\lambda^{\mathbb{R}}$. It should be noted that $\begin{bmatrix} A \\ \mathbf{I}_{m_1-n_1} \end{bmatrix}$ and $\begin{bmatrix} D \\ \mathbf{I}_{m_2-n_1} \end{bmatrix}$ are mod 2 dual characteristic maps over K_1 and K_2, respectively.

Lemma 6.3.8. *Let K be a seed and ϕ an irreducible puzzle over its J-construction* (6.3.3)*. Suppose that there are two distinct vertices v and w of $K(J)$ such that $\overline{\lambda^{\mathbb{R}}_{\phi}}(v) = \overline{\lambda^{\mathbb{R}}_{\phi}}(w)$. Then, there exists a suspended pair $\{2k-1, 2k\}$ of K such that $v = (2k-1)_s$ and $w = (2k)_t$ for some $1 \leqslant s \leqslant j_{2k-1}$ and $1 \leqslant t \leqslant j_{2k}$.*

Proof. By Corollary 6.2.10, $\{v, w\}$ is a suspended pair or a wedged edge of $K(J)$. In the first case, we are done.

Let $\{v, w\}$ be a wedged edge of $K(J)$. As in the proof of Lemma 6.3.7, there are two distinct vertices l and l' of K such that $v = l_s$, $w = l'_t$ for some $1 \leqslant s \leqslant j_l$ and $1 \leqslant t \leqslant j_{l'}$. By associativity and commutativity of the wedge and join operations, the wedged edge $\{v, w\}$ is contained in either $L(J_{q+1})$ or $\partial I_k(J_k)$ for some k. If l and l' are contained in some $\partial I_k(J_k)$, then they form a suspended pair since they are distinct vertices.

Assume that l and l' are contained in L. Let $\boldsymbol{\alpha}$ be a vertex of $G(J)$ such that its lth and l'th components are s and t, respectively. Then, $\phi(\boldsymbol{\alpha})$ is a mod 2 characteristic map over K such that $\overline{\phi(\boldsymbol{\alpha})}(l) = \overline{\phi(\boldsymbol{\alpha})}(l')$. Consider the joining representative (6.3.4) of $\phi(\boldsymbol{\alpha})$ over K by setting $K_1 = \partial I_1 * \cdots * \partial I_q$ and $K_2 = L$. Then, the mod 2 dual characteristic map $\begin{bmatrix} D \\ \mathbf{I}_{m_2-n_2} \end{bmatrix}$ over the non-suspended seed L has two repeated rows at l and l', which contradicts that non-suspended seeds only admit IDCMs. □

Theorem 6.3.9. *Let K be a seed, and ϕ a realizable puzzle over its J-construction* (6.3.3)*. Assume that there exist two distinct vertices v and w of $K(J)$ that do not form a suspended pair and that satisfy $\overline{\lambda^{\mathbb{R}}_{\phi}}(v) = \overline{\lambda^{\mathbb{R}}_{\phi}}(w)$. Then, there exists an isomorphism $f\colon K(J') \to K(J)$ for some positive integer tuple J' such that the characteristic map $\lambda^{\mathbb{R}}_{\phi} \circ f$ over $K(J')$ corresponds to a reducible puzzle over $K(J')$.*

Proof. It is enough to consider the case when ϕ is irreducible over $K(J)$. By Lemma 6.3.8, there is a suspended pair $\{2k-1, 2k\}$ of K such that $v = (2k-1)_s$ and $w = (2k)_t$. Since $\{v, w\}$ is not a suspended pair, that is $J_k \neq (1,1)$, it is a wedged edge of $K(J)$ by Corollary 6.2.10. Without loss of generality, we can assume that $j_{2k-1} \geqslant 2$. The irreducibility of ϕ ensures $\overline{\lambda^{\mathbb{R}}_{\phi}}((2k-1)_1) \neq \overline{\lambda^{\mathbb{R}}_{\phi}}((2k-1)_2)$. Consider the positive integer tuple $J' = (j'_1, \ldots, j'_m)$ such that $j'_{2k-1} = 1$, $j'_{2k} = j_{2k-1} + j_{2k} - 1$, and $j'_i = j_i$ for $i \notin \{2k-1, 2k\}$. Define a simplicial map $f\colon K(J') \to K(J)$ by

$$f(x) = \begin{cases} (2k-1)_{l-j_{2k}+1}, & \text{if } x = (2k)_l \text{ for } l > j_{2k} \\ x, & \text{otherwise.} \end{cases} \tag{6.3.5}$$

Since any J-construction of ∂I_k is the boundary of a simplex, f is an isomorphism. Then, $\lambda^{\mathbb{R}}_{\phi} \circ f$ is a mod 2 characteristic map over $K(J')$. By Remark 6.3.3, this does not correspond to an irreducible puzzle over $K(J')$. □

We call $\overline{\lambda^{\mathbb{R}}}$ *quasi-injective* if for every two distinct vertices v and w such that $\overline{\lambda^{\mathbb{R}}}(v) = \overline{\lambda^{\mathbb{R}}}(w)$, $\{v, w\}$ is a suspended pair of K.

From the above theorem, we can restate the modified toric wedge induction based on a seed as follows.

Proposition 6.3.10. *Let K be a colorable seed, and*

$$\mathcal{X}_K = \{(L, \lambda^{\mathbb{R}}) \mid L = K(J) \text{ for some } J,\ \lambda^{\mathbb{R}} \text{ is a mod 2 characteristic map over } L\}.$$

For a property $\mathcal{P}$, suppose that the following holds;

1. ***Basis step:*** *For any positive integer tuple J and any quasi-IDCM $\overline{\lambda^{\mathbb{R}}}$ over $K(J)$, $(K(J), \lambda^{\mathbb{R}})$ satisfies $\mathcal{P}$.*
2. ***Inductive step:*** *If $(L, \lambda^{\mathbb{R}}) \in \mathcal{X}_K$ satisfies $\mathcal{P}$, then so does the pair consisting of the wedge of L at v and the canonical extension of $\lambda^{\mathbb{R}}$ at v for any vertex v of L.*

Then, $\mathcal{P}$ holds on $\mathcal{X}_K$.

Even though Lemma 6.3.5 ensures that the basis step of the modified toric wedge induction is a finite problem, it is even clearer in this form. The number of suspended pairs can not exceed the Picard number, so the number of rows of a quasi-IDCM can not exceed $p + 2^p - 1$.

Remark 6.3.11. In particular, the (modified) toric wedge induction is useful when we want to see a property for all real toric spaces over PL spheres with Picard number p. By the injectivity of mod 2 dual characteristic maps over non-suspended seeds, if K is a non-suspended seed, then we have $m \leqslant 2^p - 1$, so there are finitely many non-suspended seeds with Picard number p. Since suspended seeds with Picard number p are suspensions of the (non-suspended and suspended) seeds with Picard number $p-1$, there are finitely many suspended seeds with Picard number p as well. Hence, Lemma 6.3.5 guarantees that the basis steps of the (modified) toric wedge induction based on all seeds with Picard number p can be solved in finite time.

6.4 Proof of the main theorem

For the reader's convenience, we recall the statement of the main theorem.

Theorem 6.1.2. *Let K be an $(n-1)$-dimensional PL sphere with $m \leqslant n+4$ vertices. Then, any subgroup of $\mathbb{Z}_2^m$ acting freely on $\mathbb{R}\mathcal{Z}_K$ is induced by a subtorus of T^m acting freely on $\mathcal{Z}_K$.*

Proof. Let r be the rank of a subgroup acting freely on $\mathbb{R}\mathcal{Z}_K$. We split the proof in two parts: $r \leqslant 3$ and $r = 4$.

6.4.1 The case $r \leqslant 3$

Assume that an $m \times r$ matrix S over $\mathbb{Z}_2$ represents a freely acting subgroup of $\mathbb{Z}_2^m$ on $\mathbb{R}\mathcal{Z}_K$ as (6.2.1). Define the $m \times r$ matrix $\widetilde{S}$ over $\mathbb{Z}$, all of whose entries $\widetilde{s}_{ij}$ are in $\{0,1\}$, and such that $S \equiv \widetilde{S} \bmod 2$ by a mod 2 analogue of Proposition 6.2.1. For every facet I of K, S_{I^c} has an $r \times r$ submatrix R whose determinant is $1 \in \mathbb{Z}_2$. Then the corresponding submatrix $\widetilde{R}$ of $\widetilde{S}_{I^c}$ has an odd determinant. Since the absolute value of the determinant of any square $\{0,1\}$-matrix of size $r \leqslant 3$ is in $\{0,1,2\}$, we get $\det \widetilde{R} = \pm 1 \in \mathbb{Z}$. Hence $\widetilde{S}$ defines an r-dimensional subtorus of T^m acting freely on $\mathcal{Z}_K$.

6.4.2 The case $r = 4$

In this case, it is enough to consider the case when $m = n+4$, and $r = m - n = 4$.

We apply the modified toric wedge induction of Proposition 6.3.10 with the following property $\mathcal{P}$: a pair $(L, \lambda^{\mathbb{R}})$, for L a PL sphere and $\lambda^{\mathbb{R}}$ a mod 2 characteristic map over L satisfies $\mathcal{P}$ if and only if $\lambda^{\mathbb{R}}$ has a lift. The basis step for every seed with Picard number 4 will be achieved in Section 6.5. The inductive step follows from the following lemma.

Lemma 6.4.1 (Inductive Step)**.** *For a PL sphere K, if a mod 2 characteristic map $\lambda^{\mathbb{R}}$ over K has a lift, then the canonical extension of $\lambda^{\mathbb{R}}$ at v has a lift, for every vertex v of K.*

Proof. Let K be an $(n-1)$-dimensional PL sphere on $[m]$. By relabeling the vertices and up to D-J equivalence, we may assume that the first n columns of $\lambda^{\mathbb{R}}$ and the last $m-n$ columns of $\overline{\lambda^{\mathbb{R}}}$ form an identity matrix and that the wedge operation is performed at $v = 1 \in [m]$. Let $\lambda = \widetilde{\lambda^{\mathbb{R}}}$ be a lift of $\lambda^{\mathbb{R}}$. By Proposition 6.2.4, up to D-J equivalence, we can also assume that its first n columns form the $n \times n$ identity matrix.

Set $\lambda^{\mathbb{R}} = \begin{bmatrix} \mathbf{I}_n & \begin{matrix} \overline{\lambda^{\mathbb{R}}}(1) \\ \overline{\lambda^{\mathbb{R}}}(2) \\ \vdots \\ \overline{\lambda^{\mathbb{R}}}(n) \end{matrix} \end{bmatrix}$ and its lift $\lambda = \begin{bmatrix} \mathbf{I}_n & \begin{matrix} \overline{\lambda}(1) \\ \overline{\lambda}(2) \\ \vdots \\ \overline{\lambda}(n) \end{matrix} \end{bmatrix}$. Then,

$$\Lambda^{\mathbb{R}} = \begin{bmatrix} \mathbf{I}_{n+1} & \begin{matrix} \overline{\lambda^{\mathbb{R}}}(1) \\ \overline{\lambda^{\mathbb{R}}}(1) \\ \overline{\lambda^{\mathbb{R}}}(2) \\ \vdots \\ \overline{\lambda^{\mathbb{R}}}(n) \end{matrix} \end{bmatrix} \text{ and } \Lambda = \begin{bmatrix} \mathbf{I}_{n+1} & \begin{matrix} \overline{\lambda}(1) \\ \overline{\lambda}(1) \\ \overline{\lambda}(2) \\ \vdots \\ \overline{\lambda}(n) \end{matrix} \end{bmatrix}$$

are the canonical extension of $\lambda^{\mathbb{R}}$ at 1 and its lift, respectively. ☐

Therefore, by Proposition 6.3.10, $\mathcal{P}$ holds on the set of all real toric spaces with Picard number 4. ☐

6.5 The basis step for $r = 4$

In this section, we prove that every quasi-IDCM over a PL sphere K with Picard number 4 has a lift as a mod 2 characteristic map over $\overline{K}$. Then the basis step of Proposition 6.3.10 is accomplished by Lemma 6.2.5.

Let K be a colorable seed with Picard number 4 with a quasi-IDCM $\overline{\lambda^{\mathbb{R}}}$. There are two cases:

1. $\overline{\lambda^{\mathbb{R}}}$ is not injective.
2. $\overline{\lambda^{\mathbb{R}}}$ is injective.

We start with the first case. By the definition of a quasi-injective mod 2 dual characteristic map, K is the suspension of some colorable PL sphere with Picard number 3. Thus, $K = \partial I * L(J)$ for some seed L with Picard number 3, and a positive integer tuple J. By [44], L is one of the following:

1. $L_1 = \partial P_5$,
2. $L_2 = \partial I * \partial I * \partial I$, or
3. $L_3 = \partial C^4(7)$,

where P_5 is a pentagon, and $C^4(7)$ is a 4-dimensional cyclic polytope with 7 vertices. By Example 6.2.7, if $K = L_2(J)$, then every mod 2 characteristic map over K has a lift. Since there is no seed with Picard number 2 with $n \in \{1, 3\}$, ∂P_5 and $\partial C^4(7)$ are non-suspended seeds by Proposition 6.2.12. Thus, the row repetition of $\overline{\lambda^{\mathbb{R}}}$ occurs at the unique suspended pair of K. If we set $K_1 = \partial I$ and $K_2 = L(J)$, then by (6.3.4), $\lambda^{\mathbb{R}}$ and $\overline{\lambda^{\mathbb{R}}}$ are of the form:

$$\lambda^{\mathbb{R}} = \begin{bmatrix} 1 & 1 & \mathbf{0} \\ \mathbf{0} & * & \mu^{\mathbb{R}} \end{bmatrix}, \quad \overline{\lambda^{\mathbb{R}}} = \begin{bmatrix} 1 & \mathbf{0} \\ 1 & \mathbf{0} \\ * & \multirow{2}{*}{$\overline{\mu^{\mathbb{R}}}$} \\ \mathbf{0} & \end{bmatrix},$$

where $\mu^{\mathbb{R}}$ is a mod 2 characteristic map over $L(J)$. By the definition of the join, a subset σ of the vertex set of K is a facet of K if and only if $\sigma = \{v\} \cup \tau$ for a vertex v of ∂I and a facet

τ of $L(J)$. Since the Picard number of $L(J)$ is 3, by the case $r \leqslant 3$ of Theorem 6.1.2 proved in Section 6.4.1, there is a lift $\widetilde{\mu^{\mathbb{R}}}$ of $\mu^{\mathbb{R}}$. Then the following map is a lift of $\lambda^{\mathbb{R}}$:

$$\widetilde{\lambda^{\mathbb{R}}} = \begin{bmatrix} 1 & 1 & \mathbf{0} \\ \mathbf{0} & * & \widetilde{\mu^{\mathbb{R}}} \end{bmatrix}.$$

Indeed, let v be a vertex of ∂I and τ be a facet of $L(J)$, we have $\det(\widetilde{\lambda^{\mathbb{R}}}(\{v\} \cup \tau)) = \pm \det(\widetilde{\mu^{\mathbb{R}}}(\tau)) = \pm 1$ by the non-singularity condition of $\widetilde{\mu^{\mathbb{R}}}$ for $L(J)$. Hence, this implies the non-singularity condition of $\widetilde{\lambda^{\mathbb{R}}}$ for K.

In the rest of the section, we focus on the second case where $\overline{\lambda^{\mathbb{R}}}$ is an IDCM. We prove the result in two steps. In Theorem 6.5.1, we show that when a colorable PL sphere K with Picard number 4 has less than 168 facets, then there is a representative in the D-J class of any IDCM over K that has the $\{0,1\}$-lift, hence ruling out many cases. We then use the database of Theorem 6.2.13 for finding colorable PL spheres that have at least 168 facets and algorithmically find a lift for each of them.

Theorem 6.5.1. *Let K be a PL sphere with less than* 168 *facets. Then, every injective mod* 2 *dual characteristic map over K has a lift as a mod* 2 *characteristic map over $\overline{K}$.*

Proof. We consider the *universal complex* $X(\mathbb{Z}_2^4)$, which is the binary matroid encoding the linear independence relations over $\mathbb{Z}_2$ among all every nonzero vectors in the finite set $\mathbb{Z}_2^4$, see [16, 60, 125]. More explicitly, it is the simplicial complex whose vertex set is $\mathbb{Z}_2^4 \setminus \{\mathbf{0}\}$ and whose facets are the 4-subsets I of vectors of $\mathbb{Z}_2^4$ whose determinant is one. Through direct computations, we know that it has $|GL(4,\mathbb{Z}_2)|/4! = 840$ facets, where $GL(4,\mathbb{Z}_2)$ is the general linear group of degree 4 over $\mathbb{Z}_2$. When the vectors are regarded as the $\{0,1\}$-vectors in $\mathbb{Z}^4$, 835 facets correspond to sets of 4 vectors with determinants ± 1, and the remaining 5 correspond to sets of 4 vectors with determinants ± 3, which are as follows:

$$I_1 = \left\{ \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\1\end{bmatrix} \right\}, I_2 = \left\{ \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\\1\end{bmatrix} \right\}, I_3 = \left\{ \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\\1\end{bmatrix} \right\},$$

$$I_4 = \left\{ \begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \right\}, \text{ and } I_5 = \left\{ \begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\1\end{bmatrix} \right\}.$$

Let $a_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}$, $a_2 = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}$, $a_3 = \begin{bmatrix}1\\0\\0\\1\end{bmatrix}$, and $a_4 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}$ be the four vectors in I_1. In mod 2, we can observe that

$$\begin{aligned} I_2 &= \{a_1, a_1+a_2, a_1+a_3, a_1+a_4\}, \\ I_3 &= \{a_1+a_2, a_2, a_2+a_3, a_2+a_4\}, \\ I_4 &= \{a_1+a_3, a_2+a_3, a_3, a_3+a_4\}, \text{ and} \\ I_5 &= \{a_1+a_4, a_2+a_4, a_3+a_4, a_4\}. \end{aligned}$$

Note that this combinatorial structure does not depend on the choice of I_i's and a_j's.

An element g of $GL(4, \mathbb{Z}_2)$ induces an automorphism on $X(\mathbb{Z}_2^4)$. Assume that

$$\{I_1, I_2, \ldots, I_5\} \cap \{g(I_1), g(I_2), \ldots, g(I_5)\} \neq \varnothing,$$

that is, they contain a common element $I_i = g(I_j)$. Then, by the property we discussed above, g maps the collection of the five sets I_i to itself. Thus, for any $g \in GL(4, \mathbb{Z}_2)$, there are only two possibilities:

— $\{I_1, I_2, \ldots, I_5\} \cap \{g(I_1), g(I_2), \ldots, g(I_5)\} = \varnothing$ or

— $\{I_1, I_2, \ldots, I_5\} = \{g(I_1), g(I_2), \ldots, g(I_5)\}$.

Let $\mathcal{P} = \{\{g(I_1), g(I_2), \ldots, g(I_5)\} \mid g \in GL(4, \mathbb{Z}_2)\}$ of the set of facets of $\mathcal{M}$ with $|\mathcal{P}| = 840/5 = 168$. Then any two elements of $\mathcal{P}$ does not intersect. Since every facet of $X(\mathbb{Z}_2^4)$ can be transformed into I_1 by a suitable $g \in GL(4, \mathbb{Z}_2)$, $\mathcal{P}$ is a partition of the facet set of $X(\mathbb{Z}_2^4)$. In this partition, exactly one among the 168 sets contains the set of vectors in $\mathbb{Z}$ whose determinant is ± 3.

Now, let $\overline{\lambda^{\mathbb{R}}}$ be an IDCM over K. Since $\overline{\lambda^{\mathbb{R}}}$ is injective, it induces an embedding $\iota_{\overline{\lambda^{\mathbb{R}}}}$ of $\overline{K}$ into $X(\mathbb{Z}_2^4)$ by the mod 2 non-singularity condition. If $\overline{K}$ has less than 168 facets, then there exists an element P of $\mathcal{P}$ that does not intersect the facet set of $\iota_{\overline{\lambda^{\mathbb{R}}}}\left(\overline{K}\right)$ by the reverse pigeonhole principle. There exists $g \in GL(4, \mathbb{Z}_2)$ such that $g(P) = \{I_1, I_2, \ldots, I_5\}$ by the definition of $\mathcal{P}$. For any $i = 1, 2, \ldots, 5$, the set I_i is not a facet of $\iota_{\overline{\lambda^{\mathbb{R}}}g^\top}\left(\overline{K}\right)$ since $g^{-1}(I_i) \in P$ is not a facet of $\iota_{\overline{\lambda^{\mathbb{R}}}}\left(\overline{K}\right)$. Hence, $\overline{\lambda^{\mathbb{R}}}g^\top$ has a $\{0,1\}$-lift. By Proposition 6.2.4, $\overline{\lambda^{\mathbb{R}}}$ has a lift as a mod 2 characteristic map over $\overline{K}$. □

In turn, let us consider the case where K is an $(n-1)$-dimensional PL sphere which has at least 168 facets and which supports an IDCM. By using the list of colorable seeds with Picard number 4 (as in Theorem 6.2.13), one can check that if $n < 10$, then any colorable PL sphere with Picard number 4 has less than 168 facets, hence $n \geqslant 10$. On the other hand, the condition of supporting an IDCM implies that $m \leqslant 15$, so $n \leqslant 11$. In addition, we can check whether there is an IDCM over a given PL sphere K with $10 \leqslant n \leqslant 11$ by the Garrison-Scott algorithm [75] for finding all mod 2 characteristic maps over K or the modified version introduced in [49] for finding only IDCMs, which is faster with small Picard numbers. We therefore apply the following algorithm for finding the PL spheres having at least 168 facets and which support an IDCM.

Algorithm 6.5.2.
Let ε_k be the tuple whose kth coordinate is 1 and all its other coordinates are 0.

— **Input:** a seed L.

— **Initialization:**

$\mathcal{J} \leftarrow \{(1, \ldots, 1)\}$,
$\mathcal{J}_{\text{excl}} \leftarrow \varnothing$,
$\mathcal{J}_{\text{keep}} \leftarrow \varnothing$,
$\mathcal{R} \leftarrow \varnothing$.

— **Output:** the list $\mathcal{R}$ containing the tuples of positive integers J such that $L(J)$ admits an IDCM and has 168 facets or more.

— **Procedure:**

1. $J \leftarrow \mathcal{J}[1]$, and remove J from $\mathcal{J}$.
2. If there is no IDCM over $K(J)$, then add J to $\mathcal{J}_{\text{excl}}$, and go to (1).
3. Add J to $\mathcal{J}_{\text{keep}}$.

4. If $L(J)$ has 168 facets or more, then add J to $\mathcal{R}$.
5. If $\mathcal{J} \neq \varnothing$, then go to (1).
6. If $\dim(K(J)) = 10$, then return $\mathcal{R}$.
7. Set $\mathcal{J}_{\text{keep}} \leftarrow$ the list of $J + \varepsilon_k$, for every $J \in \mathcal{J}_{\text{keep}}$ and every index k.
8. Set $\mathcal{J}_{\text{excl}} \leftarrow$ the list of $J + \varepsilon_k$, for every $J \in \mathcal{J}_{\text{excl}}$ and every index k.
9. Set $\mathcal{J} \leftarrow \mathcal{J}_{\text{keep}} \setminus \mathcal{J}_{\text{excl}}$.
10. Set $\mathcal{J}_{\text{keep}} \leftarrow \varnothing$.
11. Go to (1).

After performing the selection of Algorithm 6.5.2, only 21 PL spheres remain, each having at least 168 facets, and 88 IDCMs over them. Table 6.1 displays the number of such PL spheres and IDCMs over them.

	K			$(K, \overline{\lambda^{\mathbb{R}}})$		
n	seeds	non-seeds	total	seeds	non-seeds	total
10	11	0	11	11	0	11
11	4	6	10	5	72	77

Table 6.1 – The numbers of PL sphere K supporting an IDCM and having at least 168 facets (left) and the total number of IDCMs over them (right). Note that for $n = 11$, there is one seed that has two IDCMs, five out of the six non-seeds have 8 IDCMs and the remaining one has 32 IDCMs.

Two IDCMs $\Lambda_1^{\mathbb{R}}$ and $\Lambda_2^{\mathbb{R}}$ over a non-seed K are said to be *symmetric* if they can be expressed as $\Lambda_1^{\mathbb{R}} = \lambda_1^{\mathbb{R}} \wedge_v \lambda_2^{\mathbb{R}}$ and $\Lambda_2^{\mathbb{R}} = \lambda_2^{\mathbb{R}} \wedge_v \lambda_1^{\mathbb{R}}$ for some IDCMs $\lambda_1^{\mathbb{R}}$ and $\lambda_2^{\mathbb{R}}$. Although they are distinguished as IDCMs, the existence of their lifts are equivalent. Therefore, it is enough to consider all IDCMs up to symmetry. By reducing symmetries, there remain 22 IDCMs that we treat algorithmically; there is only one IDCM over each non-seed K of Table 6.1, and only one seed supports two IDCMs, see Table 6.2.

	$(K, \overline{\lambda^{\mathbb{R}}})$		
n	seeds	non-seeds	total
10	11	0	11
11	5	6	11

Table 6.2 – The number of IDCMs up to symmetry over the PL spheres K having at least 168 facets.

The final step is to check whether all twenty-two pairs $(K, \lambda^{\mathbb{R}})$ have $\{0, \pm 1\}$-lifts by the following simple brute-force algorithm.

Algorithm 6.5.3.

— **Input:** the cofacets CF of K and a mod 2 dual characteristic map $\overline{\lambda^{\mathbb{R}}}$ over K

— **Initialization:**

$I \leftarrow$ the list containing the coordinates of the nonzero entries of $\overline{\lambda^{\mathbb{R}}}$,
$i \leftarrow 0$.

— **Output:** a $\{0,\pm 1\}$-lift $\overline{\lambda}$ of $\overline{\lambda^{\mathbb{R}}}$ if it exists and 0 otherwise.

— **Procedure:**

1. If $i = |I|$, then return 0.
2. Set $S \leftarrow$ the list of all i-subsets of I.
3. If $S = \varnothing$, then set $i \leftarrow i+1$, and go to (1).
4. Set $s \leftarrow S[1]$, remove s from S.
5. Let $\overline{\lambda}$ be the $\{0,\pm 1\}$-matrix whose nonzero entries are the ones of $\overline{\lambda^{\mathbb{R}}}$, and which are -1 for every coordinate in s, and 1 otherwise.
6. If there is a cofacet $cf \in CF$ such that the determinant of the matrix consisting of the rows of $\overline{\lambda}$ corresponding to cf is not ± 1, then go to (3).
7. Return $\overline{\lambda}$.

After running Algorithm 6.5.3, we obtain a $\{0,\pm 1\}$-lift for every pair in Table 6.2, and therefore, combined with Theorem 6.5.1, we obtain the following theorem.

Theorem 6.5.4. *For a PL sphere K with Picard number 4, every quasi-injective mod 2 dual characteristic map over K has a lift as a mod 2 characteristic map over $\overline{K}$.*

6.6 The PL spheres with at least 168 facets which support an IDCM

For completeness, we provide the reader with the 22 pairs $(K, \overline{\lambda^{\mathbb{R}}})$, for the seeds K having at least 168 facets and $\overline{\lambda^{\mathbb{R}}}$ an IDCM over K, together with a $\{0,\pm 1\}$-lift $\overline{\lambda}$ of $\overline{\lambda^{\mathbb{R}}}$. Note that this is not in the article [41].

We label them as $K_1, \dots, K_{21}$, with K_{21} supporting two IDCMs. Moreover, we describe them by their set of minimal non-faces which are written in a concatenated way for lighter notations. For that reason, the vertices $10, 11, 12, 13, 14$, and 15 are replaced by the letters A, B, C, D, E, and F, respectively.

We split this section into three parts that cover the non-seeds, the seeds with $n = 10$, and the seeds with $n = 11$.

6.6.1 The non-seeds

Let K_1^5 be the PL sphere on $\{1, 2, \dots, 9\}$ whose minimal non-faces are

$$\{146, 168, 259, 269, 347, 349, 459, 467, 469, 689, 1258, 1278, 1357, 1378, 2357, 2358\}.$$

Then, $K_1 = K_1^5(2,2,2,1,2,1,2,2,1)$ has the following minimal non-faces

$$\{123489DE, 1234BCDE, 125689BC, 1256BCDE, 127A, 12ADE, \\ 345689BC, 345689DE, 3489F, 34AF, 567BC, 567F, 789F, 7ABC, 7AF, ADEF\},$$

after relabeling $\{1_1, 1_2, 2_1, 2_2, 3_1, 3_2, 4, 5_1, 5_2, 6, 7_1, 7_2, 8_1, 8_2, 9\}$ to $\{1, 2, \dots, 9, A, \dots, F\}$. Its f-vector is

$$(15, 105, 454, 1347, 2874, 4494, 5160, 4293, 2494, 924, 168).$$

We have:

$$\overline{\lambda^{\mathbb{R}}}^{\top} = \overline{\lambda}^{\top} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Let K_1^7 be the PL sphere on $\{1, 2, \dots, 9, A, B\}$ whose minimal non-faces are

$$\{159, 179, 459, 79A, 1278, 128B, 129B, 158B, 237A, 456B, 2368B, 3456A, 3467A, 3468A\}.$$

Then, $K_2 = K_1^7(1, 2, 2, 1, 1, 1, 2, 1, 1, 2, 1)$ has the following minimal non-faces:

$$\{1239AB, 123BF, 123CF, 17BF, 17C, 19AC, 23458BF, 23459ADE, \\ 45678DE, 45689ADE, 4568BDE, 678F, 67C, 9ACDE\},$$

after relabeling $\{1, 2_1, 2_2, 3_1, 3_2, 4, 5, 6, 7_1, 7_2, 8, 9, A_1, A_2, B\}$ to $\{1, \dots, 9, A \dots, F\}$.
Its f-vector is

$$(15, 105, 453, 1339, 2849, 4459, 5150, 4326, 2538, 946, 172).$$

We have:

$$\overline{\lambda^{\mathbb{R}}}^{\top} = \overline{\lambda}^{\top} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Let K_2^7 be the PL sphere on $\{1, 2, \dots, 9, A, B\}$ whose minimal non-faces are

$$\{179, 37A, 79A, 1259, 1278, 128B, 129B, 2459, 4569, 456B, \\ 4579, 138AB, 2368B, 3456A, 3468A, 368AB\}.$$

Then, $K_3 = K_2^7(1, 1, 1, 2, 2, 2, 1, 1, 1, 1, 2)$ has the following minimal non-faces

$$\{1267C, 12AB, 12BEF, 12CEF, 13BDEF, 1AC, 2389BEF, 24567C, 3456789D, \\ 34589BD, 389BDEF, 3AD, 456789C, 456789EF, 4567AC, ACD\},$$

after relabeling $\{1, 2, 3, 4_1, 4_2, 5_1, 5_2, 6_1, 6_2, 7, 8, 9, A, B_1, B_2\}$ to $\{1, \dots, 9, A \dots, F\}$.
Its f-vector is

$$(15, 105, 452, 1330, 2814, 4382, 5044, 4230, 2480, 924, 168).$$

We have:

$$\overline{\lambda^{\mathbb{R}}}^{\top} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix},$$

and a $\{0, \pm 1\}$-lift is

$$\overline{\lambda}^{\top} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Let K_3^7 be the PL sphere on $\{1,2,\dots,9,A,B\}$ whose minimal non-faces are

$$\{37A, 78A, 1259, 1278, 1279, 128B, 129B, 1379, 18AB, 268B,\\ 456B, 68AB, 23459, 34569, 3456A, 34579, 3468A\}.$$

Then, $K_4 = K_3^7(1,1,1,2,2,2,1,1,1,1,2)$ has the following minimal non-faces

$$\{1267C, 12AB, 12AC, 12BEF, 12CEF, 13AC, 1BDEF, 234567C, 289BEF,\\ 3456789C, 3456789D, 34567AC, 34589BD, 3AD, 456789EF, 89BDEF, ABD\},$$

after relabeling $\{1,2,3,4_1,4_2,5_1,5_2,6_1,6_2,7,8,9,A,B_1,B_2\}$ to $\{1,\dots,9,A\dots,F\}$.
Its f-vector is

$$(15, 105, 453, 1339, 2849, 4459, 5150, 4326, 2538, 946, 172).$$

We have:

$$\overline{\lambda^{\mathbb{R}}}^{\top} = \begin{bmatrix} 1&0&1&0&0&1&1&0&0&0&1&0&1&1&1\\ 0&1&1&0&0&0&0&1&1&0&1&1&0&1&1\\ 0&0&0&1&0&1&0&1&0&1&1&1&1&0&1\\ 0&0&0&0&1&0&1&0&1&1&1&1&1&1&0 \end{bmatrix},$$

and a $\{0,\pm1\}$-lift is

$$\overline{\lambda}^{\top} = \begin{bmatrix} 1&0&1&0&0&-1&1&0&0&0&1&0&1&1&1\\ 0&1&1&0&0&0&0&1&-1&0&1&1&0&1&1\\ 0&0&0&1&0&1&0&1&0&1&1&1&1&0&1\\ 0&0&0&0&1&0&1&0&1&1&1&1&1&1&0 \end{bmatrix}.$$

Let K_4^7 be the PL sphere on $\{1,2,\dots,9,A,B\}$ whose minimal non-faces are

$$\{379, 37A, 78A, 1278, 156A, 239B, 359B, 35AB, 456B, 567A,\\ 56AB, 12468, 12489, 1249B, 1468A, 23489\}.$$

Then, $K_1 = K_4^7(1,1,1,2,2,2,1,1,1,1,2)$ has the following minimal non-faces

$$\{124589B, 1245BC, 1245CEF, 12AB, 14589BD, 16789D, 2345BC, 23CEF, 367CEF,\\ 367DEF, 3AC, 3AD, 456789EF, 6789AD, 6789DEF, ABD\},$$

after relabeling $\{1,2,3,4_1,4_2,5_1,5_2,6_1,6_2,7,8,9,A,B_1,B_2\}$ to $\{1,\dots,9,A\dots,F\}$.
Its f-vector is

$$(15, 105, 452, 1330, 2816, 4396, 5084, 4290, 2530, 946, 172).$$

We have:

$$\overline{\lambda^{\mathbb{R}}}^{\top} = \overline{\lambda}^{\top} = \begin{bmatrix} 1&0&1&0&0&1&1&0&0&0&1&0&1&1&1\\ 0&1&1&0&0&0&0&1&1&0&1&1&0&1&1\\ 0&0&0&1&0&1&0&1&0&1&1&1&1&0&1\\ 0&0&0&0&1&0&1&0&1&1&1&1&1&1&0 \end{bmatrix}.$$

Let K_5^7 be the PL sphere on $\{1,2,\dots,9,A,B\}$ whose minimal non-faces are

$$\{159, 179, 459, 49A, 79A, 1278, 158B, 237A, 346A, 456B, 2368B\}.$$

Then, $K_1 = K_5^7(2,2,1,1,1,1,2,2,1,1,1)$ has the following minimal non-faces

$$\{12349ABC, 127BCF, 127D, 129AD, 3458BCF, 3459AE, 568E, 678F, 67D, 6DE, 9ADE\},$$

after relabeling $\{1_1, 1_2, 2_1, 2_2, 3, 4, 5, 6, 7_1, 7_2, 8_1, 8_2, 9, A, B\}$ to $\{1, \dots, 9, A \dots, F\}$.

Its f-vector is

$$(15, 105, 453, 1338, 2841, 4431, 5094, 4257, 2486, 924, 168).$$

We have:

$$\overline{\lambda^{\mathbb{R}}}^{\top} = \overline{\lambda}^{\top} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

6.6.2 The seeds with $n = 10$

Let K_7 be the PL sphere on $\{1, \dots, 9, A \dots, E\}$ whose minimal non-faces are

$$\{128C, 238C, 238D, 1248B, 125AC, 1458C, 145AC, 1478B, 1478C, 147AC, 2356C, 256AC, 456AC, 1378BD, 1479AB, 2356DE, 479BDE, 5679AE, 569ACE, 2369BDE, 3569ADE, 3679BDE, 369ABDE, 3789BDE\}.$$

Its f-vector is

$$(14, 91, 364, 998, 1964, 2806, 2864, 1991, 850, 170).$$

We have:

$$\overline{\lambda^{\mathbb{R}}}^{\top} = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and a $\{0, \pm 1\}$-lift is

$$\overline{\lambda}^{\top} = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let K_8 be the PL sphere on $\{1, \dots, 9, A \dots, E\}$ whose minimal non-faces are

$$\{147C, 148B, 148C, 1238D, 1245C, 125CD, 128CD, 138BD, 238AD, 38ABD, 48ABD, 12569C, 1479AB, 23568D, 23569C, 2356CD, 2356DE, 378ABE, 478ABE, 479ABE, 479BCE, 5679AE, 25679CE, 3679ABE, 369ABDE, 45679CE\}.$$

Its f-vector is

$$(14, 91, 364, 998, 1966, 2816, 2884, 2011, 860, 172).$$

We have:

$$\overline{\lambda^{\mathbb{R}}}^{\top} = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and a $\{0, \pm 1\}$-lift is

$$\overline{\lambda}^\top = \begin{bmatrix} -1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let K_9 be the PL sphere on $\{1, \dots, 9, A \dots, E\}$ whose minimal non-faces are

$$\{148B, 148C, 1238D, 128CD, 138BD, 1479B, 147AC, 238AD, 28ACD, 47ACE, 48ACD, 13469B, 13569B, 1369BD, 23568D, 23569B, 2356DE, 257ACE, 25ACDE, 278ACE, 479BCE, 5679AE, 35679BE, 45679BE\}.$$

Its f-vector is

$$(14, 91, 364, 999, 1974, 2842, 2928, 2052, 880, 176).$$

We have:

$$\overline{\lambda^{\mathbb{R}}}^\top = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and a $\{0, \pm 1\}$-lift is

$$\overline{\lambda}^\top = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let K_{10} be the PL sphere on $\{1, \dots, 9, A \dots, E\}$ whose minimal non-faces are

$$\{145C, 147C, 1238D, 1248B, 1248C, 125CD, 128CD, 1478B, 238AD, 2569C, 256CD, 4569C, 456CD, 1378BD, 1479AB, 23568D, 2356DE, 479ABE, 479BCE, 5679AE, 569ACE, 1378ABE, 2378ABE, 3569ADE, 3679ABE, 369ABDE, 3789ABE\}.$$

Its f-vector is

$$(14, 91, 364, 999, 1972, 2832, 2908, 2032, 870, 174).$$

We have:

$$\overline{\lambda^{\mathbb{R}}}^\top = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and a $\{0, \pm 1\}$-lift is

$$\overline{\lambda}^\top = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let K_{11} be the PL sphere on $\{1,\dots,9,A\dots,E\}$ whose minimal non-faces are

$$\{28AD, 58AD, 1268B, 1268D, 1346C, 1358D, 135DE, 1368D, 136CD, 136DE, 15ADE, 268AB, 278AB, 1236BC, 2369BC, 2689BC, 3457DE, 4579AE, 457ADE, 479BCE, 5789AE, 13459CE, 24579AB, 24679BC, 2479ABC, 34579CE\}.$$

Its f-vector is

$$(14, 91, 364, 999, 1972, 2832, 2908, 2032, 870, 174).$$

We have:

$$\overline{\lambda^{\mathbb{R}}}^{\top} = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and a $\{0,\pm1\}$-lift is

$$\overline{\lambda}^{\top} = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let K_{12} be the PL sphere on $\{1,\dots,9,A\dots,E\}$ whose minimal non-faces are

$$\{128D, 158D, 1248B, 15ADE, 238AD, 367AC, 368AD, 36ACD, 36ADE, 568AD, 56ADE, 1238BC, 14579E, 1457DE, 14589E, 2369BC, 236ABC, 2389BC, 238ABC, 45679E, 4567DE, 479BCE, 5679AE, 124579B, 12479BC, 23479BC\}.$$

Its f-vector is

$$(14, 91, 364, 999, 1974, 2842, 2928, 2052, 880, 176).$$

We have:

$$\overline{\lambda^{\mathbb{R}}}^{\top} = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and a $\{0,\pm1\}$-lift is

$$\overline{\lambda}^{\top} = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let K_{13} be the PL sphere on $\{1,\dots,9,A\dots,E\}$ whose minimal non-faces are

$$\{128B, 138B, 128CD, 1348C, 1369B, 148CD, 259AB, 289AB, 289AD, 28ABC, 28ACD, 47CDE, 48ACD, 13467B, 13467C, 23569B, 259ADE, 25ACDE, 5679AE, 134567E, 345679E, 34567DE, 35679BE\}.$$

Its f-vector is

$$(14, 91, 364, 999, 1972, 2832, 2908, 2032, 870, 174).$$

We have:

$$\overline{\lambda^{\mathbb{R}}}^\top = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and a $\{0, \pm 1\}$-lift is

$$\overline{\lambda}^\top = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let K_{14} be the PL sphere on $\{1, \ldots, 9, A \ldots, E\}$ whose minimal non-faces are

$$\{568E, 1358D, 135AD, 139AD, 147AC, 158CD, 158CE, 15ACD, 178CD, 178CE, 17ACD, 2356D, 3568D, 568CD, 78BCE, 2349AD, 23569B, 2468BE, 479BCE, 12479AB, 23469AB, 23469BE, 23479AB, 24679BE, 2479ABE\}.$$

Its f-vector is

$$(14, 91, 364, 1000, 1978, 2848, 2932, 2053, 880, 176).$$

We have:

$$\overline{\lambda^{\mathbb{R}}}^\top = \overline{\lambda}^\top = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let K_{15} be the PL sphere on $\{1, \ldots, 9, A \ldots, E\}$ whose minimal non-faces are

$$\{1238D, 1248B, 1248C, 128CD, 1458B, 1458C, 1478C, 147AC, 238AD, 3569B, 127ACE, 12ACDE, 1358BD, 14579B, 1479BC, 23568D, 2356DE, 236ADE, 237ACE, 23ACDE, 279ACE, 3568BD, 45679B, 45689B, 4568BD, 479ACE, 479BCE, 5679AE, 679ACE, 3679ADE, 369ABDE\}.$$

Its f-vector is

$$(14, 91, 364, 1001, 1992, 2904, 3036, 2154, 930, 186)$$

We have:

$$\overline{\lambda^{\mathbb{R}}}^\top = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and a $\{0, \pm 1\}$-lift is

$$\overline{\lambda}^\top = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let K_{16} be the PL sphere on $\{1,\dots,9,A\dots,E\}$ whose minimal non-faces are

$$\{124B, 124C, 147C, 1234D, 1235D, 125CD, 1358D, 1458C, 158CD, 178CD, 2349B, 2349D, 2356D, 178ACE, 23569B, 2479BE, 3568AD, 479BCE, 568ACD, 78ABCE, 2369ABE, 3679ABE, 369ABDE, 4679ABE, 56789AE, 5678ACE, 6789ABE\}.$$

Its f-vector is

$$(14, 91, 364, 998, 1964, 2806, 2864, 1991, 850, 170)$$

We have:

$$\overline{\lambda^{\mathbb{R}}}^{\top} = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and a $\{0,\pm1\}$-lift is

$$\overline{\lambda}^{\top} = \begin{bmatrix} -1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let K_{17} be the PL sphere on $\{1,\dots,9,A\dots,E\}$ whose minimal non-faces are

$$\{1248B, 1248C, 1358D, 1458C, 1478C, 147AC, 158CD, 178CD, 17ACE, 7ABCE, 12348D, 12479B, 1479BC, 15ACDE, 23469B, 23479B, 23489B, 23489D, 23568D, 23569B, 2356DE, 2479BE, 3568AD, 356ADE, 479BCE, 5679AE, 567ACE, 568ACD, 56ACDE, 679ABE, 2369ABE, 369ABDE\}.$$

Its f-vector is

$$(14, 91, 364, 1001, 1992, 2904, 3036, 2154, 930, 186)$$

We have:

$$\overline{\lambda^{\mathbb{R}}}^{\top} = \overline{\lambda}^{\top} = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

6.6.3 The seeds with $n = 11$

Let K_{18} be the PL sphere on $\{1,\dots,9,A\dots,F\}$ whose minimal non-faces are

$$\{246D, 24BD, 2BDE, 49BD, 9BDE, 19ABE, 28ACE, 28ADE, 2ABCE, 3467D, 179ABF, 179BDF, 2468CE, 13579BF, 13579CF, 158ACEF, 234568C, 23568AC, 345678F, 1345679F, 135678ACF\}.$$

Its f-vector is

$$(15, 105, 455, 1360, 2948, 4732, 5630, 4875, 2935, 1111, 202)$$

We have:

$$\overline{\lambda^{\mathbb{R}}}^{\top} = \overline{\lambda}^{\top} = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let K_{19} be the PL sphere $\{1, \dots, 9, A \dots, F\}$ whose minimal non-faces are

$$\{49BD, 9BDE, 147BD, 17ABF, 17BDF, 19ABE, 1ABEF, 1BDEF, 2469D, 13467D, 1357BF, 1357CF, 289ACE, 289ADE, 29ABCE, 2ABCEF, 34679D, 134567F, 158ACEF, 23568AC, 24689CE, 258ACEF, 345678D, 345678F, 2345678C, 2345689C\}.$$

Its f-vector is

$$(15, 105, 455, 1363, 2975, 4837, 5858, 5172, 3166, 1210, 220)$$

We have:

$$\overline{\lambda^{\mathbb{R}}}^{\top} = \overline{\lambda}^{\top} = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

K_{20}

Let K_{20} be the PL sphere $\{1, \dots, 9, A \dots, F\}$ whose minimal non-faces are

$$\{128CE, 12BCE, 12BDE, 19ABE, 19BDE, 2347D, 2357C, 237CE, 237DE, 247BD, 27BCE, 27BDE, 3467D, 479BD, 49BDF, 79BDE, 158ACEF, 1689ABF, 23568AC, 345678F, 4569ADF, 13568ACF, 145689AF, 15689ACF, 345689AF, 34568ACF\}$$

Then, K_{20} has f-vector

$$(15, 105, 455, 1365, 2987, 4865, 5890, 5190, 3170, 1210, 220)$$

We have:

$$\overline{\lambda^{\mathbb{R}}}^{\top} = \overline{\lambda}^{\top} = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let K_{21} be the PL sphere $\{1, \dots, 9, A \dots, F\}$ whose minimal non-faces are

$$\{12CE, 12DE, 1ACE, 29DE, 9ABE, 9BDE, ABCE, 12357C, 12457D, 1367AC, 2459DF, 368ABC, 489BDF, 689ABF, 345678F\}.$$

Its f-vector is

$$(15, 105, 455, 1358, 2933, 4683, 5538, 4767, 2856, 1078, 196)$$

It has two IDCMs which are their own $\{0, 1\}$-lifts as follows:

$$\overline{\lambda_1^{\mathbb{R}}}^{\top} = \overline{\lambda_1}^{\top} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and

$$\overline{\lambda_2^{\mathbb{R}}}^{\top} = \overline{\lambda_2}^{\top} = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

* *
*

7 Perspectives

> *"Now is no time to think of what you do not have. Think of what you can do with what there is."*
>
> *Santiago — The Old Man and the Sea*

In the future, I aim to:

— Classify integer polytopes with small Picard number and study their integer decomposition properties.

— Go deeper in the study of totally equimodular matrices and box-TDI systems, for helping detecting and optimizing over such systems.

— Strengthen the links between toric topology and the combinatorial objects appearing in this theory such as binary matroids and operads.

— Create better access and easier use of the algorithms I have programmed by contributing to mathematical software such as Julia and creating an online database of toric manifolds and PL spheres.

In a recent preprint with Suyoung Choi and Hyeontae Jang [40], we leverage the database of toric-colorable seeds with Picard number 4 in [39] (or Chapter 4) to classify toric manifolds with this Picard number. I wish to use this database for dealing with Oda's conjecture concerning Delzant polytopes with Picard number 4, or at least those which are box-TDI, see 7.2.3. I also aim to find an even better algorithm for being able to find the database of toric-colorable seeds with Picard number 5, as it is the next step coming after Picard number 4, see 7.1.2. Concerning totally equimodular matrices, I would like to complete Theorem 3.5 by showing the non-existence of thick te-interlaces of size greater than 6, see 7.2.1. Moreover, finding the complexity for a decomposition algorithm of full row rank totally equimodular matrices would be very interesting, as well as a decomposition theorem for the general case of totally equimodular matrices, see 7.2.2. I wish to include more matroid theory into real toric geometry as I think the mod 2 puzzle algorithm [49] (or Chapter 5) could be improved using binary matroids, see 7.1.3. I wish to continue the study of operads on simplicial complexes 7.3. Finally, I have been programming many algorithms for toric geometry that are available in my Github. However, I believe that in order to make them more user-friendly and accessible, I should include them in the OSCAR package 7.4 of Julia. Moreover, I wish to develop an online database of toric manifolds and PL spheres, that would be accessible to anyone and would serve researchers working in toric topology and combinatorics as a sandbox to test their conjectures on.

7.1 Toric topology

7.1.1 Classification of toric manifolds with Picard number 4 (short term, ongoing)

By [39], we have the list of every toric-colorable seeds with Picard number 4. In [40], we characterize every toric manifolds with Picard number 4, there are 59 seeds that are fan-giving. One further work would be to find which fans are projective among them, that is, which fans are normal fans of Delzant polytopes.

There are two ways of tackling this problem, either by using Shephard diagrams [126], as suggested in [44], or a criterion on primitive collections of the pair (K, λ) given by Batyrev in [17]. Then, a toric wedge induction on projectivity should allow us to conclude. The main issue that occurs here is that some complete non-singular fans obtained in [40] have indeterminates in their characteristic maps, and hence there are infinitely many cases to deal with.

7.1.2 Enumeration of toric colorable seeds with Picard number 5 (mid term)

This work builds on the success of the Picard number $p = 4$ case. Extending the method used for $p = 4$ to the full case $p = 5$ poses significant challenges. For $p = 5$, the PL spheres are 25-dimensional with up to 31 vertices, resulting in $\binom{31}{26} = 169,911$ facets per PL sphere K, and therefore there are $2^{169,911}$ potential pure simplicial complexes in this extreme case.

A direct application of the GPU algorithm developed for $p = 4$ is not adapted. For instance, the algorithm struggled with the $p = 4$ case on 15 vertices, with an estimated computation time of one year. For that extreme case, we employed an inductive approach, constructing toric colorable seeds of dimension n from those of dimension $n - 1$. This tactic could, in principle, be extended to $p = 5$. However, it does not guarantee that all toric colorable seeds with Picard number p can be constructed inductively.

An alternative approach might focus on the fact that the PL sphere is a sub-complex of a co-simple binary matroid, disregarding the "seed" property. This seems promising, though further exploration is needed. Additionally, Choi and Jang's recent preprint [37] suggests another direction: applying stellar subdivision to specific faces of wedged seeds to generate seeds of greater Picard numbers.

Another improvement involves reducing redundancy by considering only one representative from each isomorphism class of binary matroids. In [39], we fixed a basis of the matroid to align with the facet $\{1, \ldots, n\}$ of the PL sphere K. However, this approach inadvertently increased the number of cases input to the GPU algorithm.

Finally, while the resulting PL spheres would be of intrinsic interest in toric topology, addressing the massive enumeration problem for $p = 5$ would demand innovative high-scale parallel programming solutions. Although our current algorithm already exploits data parallelism by enumerating mod-2 sums of binary vectors in chunks, further optimization is necessary. Collaboration with experts in parallel programming architectures could be a pivotal step in tackling this challenge.

7.1.3 Binary matroids and real toric topology (long term)

In real toric topology, we consider mod 2 characteristic pairs $(K, \lambda^{\mathbb{R}})$, where $\lambda^{\mathbb{R}}$ embodies a free action of a real torus on the real moment-angle complex associated with K. However, very few toric topologists have considered the underlying binary matroid $M_{\lambda^{\mathbb{R}}}$ associated with $\lambda^{\mathbb{R}}$, even though the mod 2 non-singularity condition implies that the facets of K are bases of $M_{\lambda^{\mathbb{R}}}$.

The only related articles are very recent and focus on the study of universal complexes in toric topology [16] or the mod p Buchstaber invariant [15].

I believe that the paradigm shift from a pair $(K, \lambda^{\mathbb{R}})$ to an inclusion $K \subseteq M_{\lambda^{\mathbb{R}}}$ is promising. This perspective is motivated by several factors. First, I noticed that both the formulas for the $\mathbb{Z}_2$-cohomology ring [60, Theorem] and the integral cohomology groups of real toric manifolds [28] can be rewritten in terms of co-circuits of $M_{\lambda^{\mathbb{R}}}$. Second, I was not entirely satisfied with the puzzle algorithm in [49], as the computations for explicit realizations of the binary matroid required a basis change. Therefore, I believe that working coordinate-free will enhance this puzzle algorithm.

I have already begun a "translation job" between the vocabulary of matroid theory and simplicial complexes. For example, the join of two matroids corresponds to their 1-sum, and the wedge of a matroid at an element v is a series extension. The next step is to understand the binary matroid $M_{\lambda_1 \wedge_v \lambda_2}$ derived from M_{λ_1} and M_{λ_2}. Specifically, I aim to describe its bases, circuits, and flats.

7.2 Combinatorial optimization

7.2.1 Proof of the nonexistence of te-interlaces of size greater than 6 (short term, ongoing)

In order to complete Theorem 3.6.6, we need to show that there are no te-interlaces of size greater than 6. From [35], this is equivalent to proving that there are no complement minimally non-totally unimodular matrices of size greater than 5. By Cornuéjols [56], we know that minimally non-totally unimodular matrices are either incidence matrices of unbalanced holes or balanced matrices.

The only $0, 1$ matrices of unbalanced holes are those corresponding to odd holes, and we proved with Patrick Chervet and Roland Grappe that they are not complement minimally non-totally unimodular when their size exceeds 5. Thus, it remains to prove that $0, 1$ balanced minimally non-totally unimodular matrices of size greater than 5 are not complement minimally non-totally unimodular.

I have identified two directions to tackle this problem. The first approach, more related to combinatorial optimization, would leverage the extensive results on the decomposition of balanced matrices and their associated TDI systems [113, 51, 50, 52] to analyze the effects of complement operations. However, this direction has not been particularly fruitful so far.

The second approach involves using the decomposition theorem for complement totally unimodular matrices provided by Truemper in [133, Construction 12.5.11]. Complement totally unimodular matrices are constructed starting from one of four "seed" matrices and applying a sequence of operations of three types. Importantly, every submatrix of size $(n-1)$ of a complement minimally non-totally unimodular matrix of size n is an invertible complement totally unimodular matrix, which can be derived from this decomposition theorem.

Conversely, as shown by Camion [30], every minimally non-totally unimodular matrix has an even number of nonzero entries in each row and column. Therefore, starting from any invertible complement totally unimodular matrix U of size $(n-1)$, there exists a unique way to construct a candidate matrix of size n that satisfies the complement minimally non-totally unimodularity condition. This is done by appending an appropriate row and column to U that respect Camion's condition.

Moreover, by [133, Corollary 12.5.18], U originates from only two out of the four seeds. For one of these cases, we end up in the complement orbit of matrices corresponding to odd holes. Thus, the remaining task is to address the matrices U that are derived from the second seed.

7.2.2 Further works on totally equimodular matrices (mid term)

First, together with Roland Grappe and Patrick Chervet, we aim to establish a decomposition theorem for any totally equimodular matrix M, thereby extending Theorem 3.5.6. We select a subset of the rows of M to form a basis of its row space, obtaining a full row rank totally equimodular submatrix $\overline{M}$, which we then decompose using Theorem 3.4. Each remaining row of M is expressed as a linear combination of the rows of $\overline{M}$. The task then becomes determining the conditions on these additional rows that preserve the total equimodularity of M. A natural starting point is to study the effect of appending a single dependent row to each type of te-bricks.

Secondly, Seymour [124] showed that recognizing totally unimodular matrices can be done in polynomial time. By a Theorem of Hoffman and Kruskal [91], any associated system is box-TDI, a critical property for integer programming. Completing the study of the complexity of recognizing and decomposing a totally equimodular matrix, as suggested in [79], would provide the polyhedral counterpart.

Thirdly, we can investigate the cones generated by any totally equimodular set. These cones are of particular interest because they correspond to cones in the normal fan of some box-TDI polyhedra. Identifying their Hilbert basis is instrumental in describing a TDI system. The solved case of te-cones is especially useful since their Hilbert basis is known. In fact, any general totally equimodular cone C can be covered by its simplicial cones, which are te-cones. As a result, the Hilbert basis of C is included in the union of the Hilbert bases of the cones in this cover. This observation may facilitate the derivation of a TDI system associated with any system described by a totally equimodular matrix.

7.2.3 Integer decomposition property of Delzant polytopes with a few vertices (mid term, ongoing)

I aim to focus on Oda's problem for polytopes with a few vertices. The first approach would be to attempt to reprove the result of Robins [116] for Picard numbers $\leqslant 3$, using the toric wedge induction. I am optimistic that this new approach will require fewer computations. If successful, this method could be extended to address the unsolved case with Picard number 4, by applying a toric wedge induction starting from any lattice Delzant polytope whose normal fan is complete, non-singular, and has Picard number 4. The latter being, hopefully, obtained as a result of my joint work Suyoung Choi and Hyeontae Jang in my Research project 7.1.1.

The first challenge is that, while there is a finite number of seeds K for the pairs (K, λ), there are infinitely many λ's that yield a complete, non-singular fan. The second challenge is that, given such a fan Σ, there are infinitely many integer realizations of polytopes having Σ as their normal fan.

To address the first challenge, we propose focusing on box-TDI Delzant polytopes, which is the subject of an ongoing joint project with Roland Grappe. The fans of these polytopes are composed of totally unimodular cones, ensuring that the characteristic maps are $0,\pm 1$ and, thus, of finite quantity.

7.3 Operads on simplicial complexes (mid term, ongoing)

In a joint research with my Ph.D. co-advisor, Bruno Vallette, we are exploring operads on simplicial complexes. This topic is of particular interest both in operad theory and toric topology. So far, the only set theoric operads that have been studied are operads whose operations with n inputs are encoded by subsets of a n-set [103]. A simplicial complexe, however, is a set of sets, and operads on simplicial complexes delve even deeper: the operations with n inputs are encoded by simplicial complexes on n-vertices. There are two primary types of operads on simplicial complexes: the substitution or the composition of simplicial complexes. Bruno and I discovered that these operads have a very general context for algebras: they act on the set of morphisms in any small monoidal category with co-products. Our study brings together several examples of polyhedral product-like structures found in the literature, including the polyhedral wedge product of topological spaces [13], the polyhedral join product [136] of simplicial complexes or the polyhedral product in motivic homotopy theory of [92]. A more in-depth exploration of operad theory within the context of polyhedral products, such as finding the indecomposables and generators of the operad, as well as relations between its elements, would be highly valuable for both the operad and polyhedral product communities. In fact, Jelena Grbić, Stephen Theriault, and Anthony Bahri, experts in the polyhedral product community, whom I met at the Fields institute in the summer of 2024, suggested me to dig in this very subject.

7.4 Contributing to mathematical software (long term)

In all my research, I tend to use a significant amount of computer programming. I maintain an up-to-date GitHub repository containing all the programming code related to my research, though I am aware that much of it is not user-friendly. The majority of this code pertains to toric topology, and I am aiming to integrate my new algorithms into the Julia package `OSCAR` [111, 62]. Specifically, I wish to develop the real toric topology branch of this package.

For instance, I plan to implement the computation of $\mathbb{Z}_2$-cohomology rings of real toric manifolds as outlined in [60]. This requires an efficient method for computing Gröbner bases for ideals of polynomial rings with $\mathbb{Z}_2$ coefficients.

Finally, I aim to implement the puzzle algorithm, which will require adding the J-construction on simplicial complexes to the package as well.

* *
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My work

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